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A Generalization of NTHK-PELL Sequence and their Identities

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ABSTRACT

In this paper, We introduced the Generalized n^{th}k-Pell Sequence$\{g_{k,n}\}_{n\in\mathbb{N}}$ by changing the initial conditions. We obtain the Binet's formulas and Generating functions and some identitiesfor this Sequence. In addition, we derive some sum formulas, using the Binet's formula of Generalized n^{th}k-Pell Sequence.

***Keywords**: k-Fibonacci Sequence, k-Lucas Sequence, k-Pell Sequence, k-Pell Lucas Sequences.*
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Introduction

The n^{th} Fibonacci sequence$\{F_n\}_{n\in\mathbb{N}}$is defined by$F_0 = 0, F_1 = 1$and $F_n = F_{n-1} + F_{n-2},\ n \geq 2$and n^{th} Lucas Sequence$\{L_n\}_{n\in\mathbb{N}}$ is defined by$L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2},\ n \geq 2$[17]. Many authors have generalized the Fibonacci sequence by changing the initial conditions or changing the recurrence relations somewhat. Example of such variations include the Generalized Fibonacci sequence, Jacobsthal sequence, Tribonacci sequence, Pell, Pell-Lucas, Modified Pell Sequences (see [3, 4, 11]). The Pell Sequence $\{P_n\}_{n\in\mathbb{N}}$is defined by$P_n = 2P_{n-1} + P_{n-2},\ n \geq 2$ with initial terms $P_0 = 2, P_1 = 1$ and if we change the values of the initial terms $PL_0 = 2, PL_1 = 2$ and $q_0 = 1, q_1 = 1$ they are called the Pell-Lucas and Modified Pell Sequences, respectively. Binet's formula for the Pell Sequence $\{P_n\}_{n\in\mathbb{N}}$and Pell-Lucas numbers $\{PL_n\}_{n\in\mathbb{N}}$ are [18].

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and$PL_n = \alpha^n + \beta^n$

respectively, where $\alpha = 1 + \sqrt{2}, \beta = 1 - \sqrt{2}$ are the roots of the characteristic equation x^2-2x-1=0 andAlso, we obtain that $\alpha + \beta = 2, \alpha - \beta = 2\sqrt{2}, \alpha\beta = -1$.

Preliminaries

Falcon and Plaza [14]introduced the k-Fibonacci Sequence, which is one of the generalizations of the Fibonacci sequence. The n^{th}*k*-Fibonacci Sequence $\{F_{k,n}\}_{n\in\mathbb{N}}$for any positive real number *k*, is defined by

$F_{k,n} = kF_{k,n-1} + F_{k,n-2},\ F_{k,0} = 0, F_{k,1} = 1, n \geq 2$ (1)

thenn^{th}*k*-Lucas Sequence $\{L_{k,n}\}_{n\in\mathbb{N}}$for any positive real number *k*,is defined by

$L_{k,n} = kL_{k,n-1} + L_{k,n-2},\ L_{k,0} = 2, L_{k,1} = k, n \geq 2$ (2)

Catarino [6] defined k-Pell sequence by the recurrence relation: The n^{th}*k*-Pell Sequence $\{P_n\}_{n\in\mathbb{N}}$for any positive real number *k*,is defined by

$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, P_{k,0} = 0, P_{k,1} = 1\ n \geq 2$ (3)

Binet's formula for the n^{th}k−Pell Sequence is [6]

$P_{k,r} = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2}$ (4)

Where $r_1 = 1 + \sqrt{1+k}, r_2 = 1 - \sqrt{1+k}$ are the roots of the characteristic equation$x^2 - 2x - k = 0$.

Also, the k-Pell Lucas$\{Q_n\}_{n\in\mathbb{N}}$and Modified k-Pell $\{q_n\}_{n\in\mathbb{N}} n \geq 2$sequence are defined by: $Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2}$, $Q_{k,0} = 2, Q_{k,1} = 2$,

and

$$q_{k,n} = 2q_{k,n-1} + kq_{k,n-2}, q_{k,0} = 1, q_{k,1} = 1\ n \geq 2.$$

respectively. For more detail about these sequences see works of Catarino and Vasco [6–8].

In this paper, we introduced the Generalized n^{th}*k*-Pell Sequence by changing the initial conditions. Also, we obtain the Binet's formulas and Generating functions and some identities for this Sequence. In addition, we derive some sum formulas, using the Binet's formula of Generalizedn^{th}*k*-Pell Sequence.

the Generalized n^{th}*k*-pell Sequence

In this section, we introduced the Generalized n^{th}*k*-Pell Sequence$\{g_{k,n}\}_{n\in\mathbb{N}}$. For this Sequence,we derive the Binet's formulas, Generating functions and certain identities.

Definition 3.1. For$k \geq 1$ $n \geq 0$, the Generalized n^{th}*k*-Pell sequence $\{g_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by

$$g_{k,n} = 2g_{k,n-1} + kg_{k,n-2},$$

$g_{k,0} = 2, g_{k,1} = k - 1\ n \geq 2$ (5)

Here, the initial conditions of the *k*-Fibonacci sequence and *k*-Lucas sequence are subtracted to from $g_{k,0}$ and $g_{k,1}$ respectively. $g_{k,0} = L_{k,0} - F_{k,0} = 2$, $g_{k,1} = L_{k,1} - F_{k,1} = k - 1$ The few terms of the Sequence $g_{k,n}$ are

$$2,\ k-1, 4k-2, k^2+7k-4,\ 6k^2+12k-8, \dots\dots$$

The explicit formula for the n^{th} term of the Generalized*k*-Pell sequence is now found: Consider the characteristic equation associated with the recurrence relation (5),

$x^2 - 2x - k = 0$ (6)

With two distinct roots α and β of the equation (6) are $\alpha = 1 + \sqrt{1+k}, \beta = 1 - \sqrt{1+k}$, $k \geq 1$. Since$\sqrt{1+k} > 1$ then $\beta < 0$ so, $\beta < 0 < \alpha$. Note that since $k \geq 1$ then $\alpha + \beta = 2, \alpha - \beta = 2\sqrt{1+k}, \alpha\beta = -k$.

Theorem 3.1 The Binet's formula for the generalized n^{th}*k*-Pell Sequence $g_{k,n}$is as follows:

$g_{k,n} = \frac{\alpha^n(k-2\beta-1)}{\alpha-\beta} - \frac{\beta^n(k-2\alpha-1)}{\alpha-\beta}$ (7)

Whereα, β are the roots of (6).

*Proof.*The$\alpha = 1 + \sqrt{1+k}$and $\beta = 1 - \sqrt{1+k}$,are the roots of equation (6), the sequence$g_{k,n} = C_1\alpha^n + C_2\beta^n$ (8)

is the solution of the equation (5). We derive the distinct values $C_1 = \frac{(k-2\beta-1)}{\alpha-\beta}$and $C_2 = \frac{(k-2\alpha-1)}{\alpha-\beta}$ by giving*n* the values *n = 0* and *n = 1* and solving this system of linear equations, Now using equation (8), we obtain

$$g_{k,n} = \frac{\alpha^n(k-2\beta-1)}{\alpha-\beta} - \frac{\beta^n(k-2\alpha-1)}{\alpha-\beta}.$$

Theorem 3.2 The Generating functions for the generalized n^{th}*k*-Pell Sequence is as follows:

$G(x) = \sum_{n=0}^{\infty} g_{k,n}x^n = \frac{2+kx-5x}{1-2x-kx^2}$ (9)

Proof. The Generating function$G(x)$ is defined as follow:

$$G(x) = \sum_{n=0}^{\infty} g_{k,n}x^n = g_{k,0} + g_{k,1}x + g_{k,2}x^2 + g_{k,3}x^3 + + \cdots + g_{k,n} + \cdots$$

Using the initial conditions, we get

$$G(x) = 2 + (k-1)x + \sum_{n=2}^{\infty} g_{k,n}x^n$$

$$= 2 + (k-1)x + 2\sum_{n=2}^{\infty} g_{k,n-1}x^n + k\sum_{n=2}^{\infty} g_{k,n-2}x^n$$

$$= 2 + kx - 5x + 2x\sum_{n=0}^{\infty} g_{k,n}x^n + kx^2\sum_{n=0}^{\infty} g_{k,n}x^n$$

$$G(x)(1 - 2x - kx^2) = 2 + kx - 5x$$

and hence the Generalized n^{th}*k*-Pell Sequence's Generating functioncan be expressed as $G(x) = \sum_{n=0}^{\infty} g_{k,n}x^n = \frac{2+kx-5x}{1-2x-kx^2}$

Theorem 3.3.(Catalan's identity).Let$g_{k,n}$ be the Generalized n^{th}*k*-Pell Sequence for n ≥ 1, we get

$$g_{k,n-r}g_{k,n+r} - {g_{k,n}}^2 = (-1)^{n-r+1}k^{n-r}{P_{k,r}}^2[k^2 - 10k + 5]$$

Where$P_{k,r}$ is *k*-Pell Sequence.

*Proof.*Applying $\alpha\beta = -k$ and Binet's formula (7)

$$g_{k,n-r}g_{k,n+r} - {g_{k,n}}^2 = \frac{\begin{array}{c}[\alpha^{n-r}(k-2\beta-1)-\beta^{n-r}(k-2\alpha-1)] \\ [\alpha^{n+r}(k-2\beta-1)-\beta^{n+r}(k-2\alpha-1)]- \\ [\alpha^{n}(k-2\beta-1)-\beta^{n}(k-2\alpha-1)]^2\end{array}}{(\alpha-\beta)^2}$$

$$= \frac{(k-2\beta-1)(k-2\alpha-1)}{(\alpha-\beta)^2}[2\alpha^n\beta^n - \alpha^{n-r}\beta^{n+r} - \alpha^{n+r}\beta^{n-r}]$$

$$= (-1)^{n-r+1}k^{n-r}[k^2 - 10k + 5]\frac{\alpha^n - \beta^n}{\alpha - \beta}$$

$$= (-1)^{n-r+1}k^{n-r}[k^2 - 10k + 5]{P_{k,r}}^2$$

Using(4) the Binet's formula of *k*-Pell Sequence.

Theorem 3.4 (Cassini's identity). Let $g_{k,n}$ be the Generalized n^{th}*k*-Pell Sequence forn ≥ 1, we get

$$g_{k,n-1}g_{k,n+1} - {g_{k,n}}^2 = (-1)^n k^{n-1}[k^2 - 10k + 5]$$

Where $P_{k,r}$ is *k*-Pell Sequence.

*Proof.*Put r = 1 in Catalan's identity obtained, we get the cassini's identity.

Theorem 3.5 (d'Ocagne's identity).If*m* and *n* are positive integers, and *m* > *n* then

$$g_{k,m}g_{k,n+1} - g_{k,m+1}g_{k,n} = (-1)^n k^n P_{k,m-n}[k^2 - 10k + 5]$$

where$P_{k,m-n}$ is k−Pell Sequence.

*Proof.*Applying $\alpha\beta = -k$ and Binet's formula (7)

$$g_{k,m}g_{k,n+1} - g_{k,m+1}g_{k,n} = \frac{\begin{array}{c}(k-2\beta-1)(k-2\alpha-1) \\ (\alpha-\beta)(\alpha^m\beta^n - \alpha^n\beta^m)\end{array}}{(\alpha-\beta)^2}$$

$$= (-k)^n[k^2 - 10k + 5]\frac{\alpha^m\beta^n - \alpha^n\beta^m}{\alpha - \beta}$$

$$= (-1)^n k^n P_{k,m-n}[k^2 - 10k + 5]$$

using (4) the Binet's *k*-Pell Sequence formula.

Theorem 3.6 For positive integers $k = 1,2,3,..$ and integer $n \geq 0$, $\lim_{n\to\infty}\frac{g_{k,n+1}}{g_{k,n}} = \alpha$ which is a root of the equation $x^2 - 2x - k = 0$.

Proof.

$$\lim_{n\to\infty}\frac{g_{k,n+1}}{g_{k,n}}=\lim_{n\to\infty}\frac{\dfrac{\alpha^{n+1}(k-2\beta-1)-\beta^{n+1}}{(k-2\alpha-1)}}{\dfrac{\alpha^{n}(k-2\beta-1)-\beta^{n}}{(k-2\alpha-1)}}$$

$$=\lim_{n\to\infty}\frac{\alpha^{n+1}\left[1-\frac{\beta^{n+1}(k-2\alpha-1)}{\alpha^{n+1}(k-2\beta-1)}\right]}{\alpha^{n}\left[1-\frac{\beta^{n+1}(k-2\alpha-1)}{\alpha^{n+1}(k-2\beta-1)}\right]}$$

$$=\lim_{n\to\infty}\frac{\alpha\left[1-\frac{\beta^{n+1}(k-2\alpha-1)}{\alpha^{n+1}(k-2\beta-1)}\right]}{\left[1-\frac{\beta^{n+1}(k-2\alpha-1)}{\alpha^{n+1}(k-2\beta-1)}\right]}$$

since$\left|\frac{\beta}{\alpha}\right|<1$then$\lim_{n\to\infty}(\frac{\beta}{\alpha})^n=0$we get,

$\lim_{n\to\infty}\frac{g_{k,n+1}}{g_{k,n}}=\alpha.$

The following are some properties of Generating functions for the Generalizedn^{th}*k*−Pell Sequence.

Theorem 3.7 Let *m* and *n* be the positive integers and $g_{k,n}$Generalizedn^{th}*k*−Pell Sequence. The following equalities are valid:

1. $\sum_{n=0}^{\infty} g_{k,n+1}x^n=\frac{k+2kx-1}{1-2x-kx^2}$
2. $\sum_{n=0}^{\infty} g_{k,mn}x^n=\frac{2+P_{k,m}(kx-x)-2xP_{k,m+1}}{1-x(\alpha^m+\beta^m)+(-1)^m k^m x^2}$
3. $\sum_{n=0}^{\infty} g_{k,mn+r}x^n=\frac{\begin{pmatrix}P_{k,r}(k-1)-(-1)^m k^m\\ (k-1)P_{k,r-m}+2kP_{k,r-1}\\ -2x(-1)^{m+1}k^{m+1}P_{k,r-m-1}\end{pmatrix}}{1-x(\alpha^m+\beta^m)+(-1)^m k^m x^2}$
4. $\sum_{n=0}^{\infty} g_{k,2n}x^n=\frac{2-10x}{1-4-2kxx+k^2x^2}$
5. $\sum_{n=0}^{\infty} g_{k,2n+1}x^n=\frac{x+k+4xk-k^2x}{1-4-2kxx+k^2x^2}$

Where $P_{k,r}$ is *k*-Pell Sequence.

Proof. 1. By applying Binet's formula (7)

$$\sum_{n=0}^{\infty} g_{k,n+1}x^n=\sum_{n=0}^{\infty}\left[\frac{\dfrac{\alpha^{n+1}(k-2\beta-1)-\beta^{n+1}}{(k-2\alpha-1)}}{\alpha-\beta}\right]x^n$$

$$=\left[\frac{\alpha(k-2\beta-1)}{\alpha-\beta}\sum_{n=0}^{\infty}\alpha^n x^n-\frac{\beta(k-2\alpha-1)}{\alpha-\beta}\sum_{n=0}^{\infty}\beta^n x^n\right]$$

$$=\frac{1}{(\alpha-\beta)}\left[\frac{\alpha(k-2\beta-1)}{(1-\alpha x)}-\frac{\beta(k-2\alpha-1)}{(1-\beta x)}\right]$$

$$=\frac{k(\alpha-\beta)+2kx(\alpha-\beta)-(\alpha-\beta)}{(\alpha-\beta)(1-x(\alpha+\beta)-kx^2)}$$

$$=\frac{k+2kx-1}{(1-2x-kx^2)}$$

Similarly, by using of Binet's formula given in (7), 2., 3., 4., 5. can be shown.

Theorem 3.8 For arbitrary integers*m, n, j ≥ 1* and *j ≥ m + 1*, we get:

1. $\sum_{i=0}^{n} g_{k,mi+j}x^n=\frac{\begin{pmatrix}(k-1)[(-1)^m k^m\\ (P_{k,mn+j}-P_{k,j-m})\\ -P_{k,mn+m+j}+P_{k,j}]+\\ 2k[(-1)^m k^m(P_{k,mn+j-1}-\\ P_{k,j-m-1})-P_{k,mn+m+j-1}\\ +P_{k,j-1}\end{pmatrix}}{1-(\alpha^m+\beta^m)+(-1)^m k^m}$

2. $\sum_{i=0}^{n} g_{k,mi} x^n = \dfrac{\begin{pmatrix} (k-1)[(-1)^m k^m \\ P_{k,mn} - P_{k,m} - P_{k,mn+m}] \\ 2[1+(-1)^m k^{m+1} P_{k,mn-1} \\ -P_{k,m+1} - P_{k,mn+m-1} \end{pmatrix}}{1-(\alpha^m+\beta^m)+(-1)^m k^m}$

3. $\sum_{i=0}^{n} g_{k,i+j} x^n = \dfrac{\begin{pmatrix} (k^2+k)[P_{k,n+j} - P_{k,j-1}] \\ +(k+1)[P_{k,n+j+1} - P_{k,j}] \\ +2k^2[P_{k,n+j-1} - P_{k,j-2}] \end{pmatrix}}{k+1}$

Where $P_{k,r}$ is k−Pell Sequence.

Proof. 1. Applying $\alpha\beta = -k$ and Binet's formula (7)

$$\sum_{i=0}^{n} g_{k,mi+j} = \sum_{i=0}^{n} \left[\frac{\alpha^{mi+j}(k-2\beta-1) - \beta^{mi+j} (k-2\alpha-1)}{\alpha - \beta} \right]$$

$$= \left[\frac{\alpha^j (k-2\beta-1)}{\alpha-\beta} \sum_{i=0}^{n} (\alpha^m)^i - \frac{\beta^j (k-2\alpha-1)}{\alpha-\beta} \sum_{i=0}^{n} (\beta^m)^i \right]$$

$$= \left[\frac{\alpha^j (k-2\beta-1)(\alpha^{mn+m}-1)}{(\alpha^m-1)(\alpha-\beta)} - \frac{\beta^j (k-2\alpha-1)(\beta^{mn+m}-1)}{(\beta^m-1)(\alpha-\beta)} \right]$$

$$= \frac{\begin{pmatrix} (k-1)[(-1)^m k^m \\ (P_{k,mn+j} - P_{k,j-m}) \\ -P_{k,mn+m+j} + P_{k,j}] + \\ 2k[(-1)^m k^m (P_{k,mn+j-1} - \\ P_{k,j-m-1}) - P_{k,mn+m+j-1} \\ +P_{k,j-1} \end{pmatrix}}{1-(\alpha^m+\beta^m)+(-1)^m k^m}$$

Using (4) the *k*-Pell Sequence Binet's formula.Similarly,2. and 3.can be demonstrated by applying Binet's formula, which is provided in (6).

Conclusion

A new Generalizedn^{th}*k*-Pell Sequence has been presented and examined. This sequencegenerating functions and other attributes are inferred. In the future, we think that it can be found many more properties of these Generalized n^{th} k-Pell Sequence.

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