International Journal of Innovations & Research Analysis (IJIRA)
ISSN :2583-0295, Impact Factor: 5.449, Volume 03, No. 04(II), October- December, 2023, pp 125-131

A MATHEMATICAL AND GRAPHICAL INVESTIGATION INTO A FEEDBACK QUEUEING MODEL WITH THREE SERVERS CONNECTED TO A COMMON FOURTH SERVER AND A ONCE-REVISIT FEATURE

Surender Kumar*
Vinod Bhatia**
Nidhi Sharma***

ABSTRACT

The present paper deals with the study of mathematical and graphical study of mean queue length of the system. The Queuing system has four servers for the service of customers. A customer, after getting service from first server goes to second or third or fourth server for service depending upon the need of service. She/he is allowed to revisit but not more than once. The arrival and service pattern are assumed to follow the Poisson process. The Mean Queue Length of the system has been calculated by solving the steady state equations by using the generating function technique.

Keywords: *Feedback, Queuing System, Poisson Process, Four Server, Mean Queue Length.*

Introduction

The literature on queuing theory offers many reports of extensive effort. Many scholars have examined queuing models with different service types, multiclass services, or multi-servers, including [1, 3, 4, 6, 14]. Additionally, it's possible that a customer will need to repeatedly join several service lines before a server completes their task to their satisfaction. These circumstances, which can be seen in manufacturing, healthcare, and other settings, gave rise to feedback in queuing systems and, as a result, enhanced the literature on queuing theory by introducing the idea of researching queuing systems with feedback [2, 5, 7–13, 15]. A hierarchical feedback queuing approach with three servers was also covered by Kumar and Taneja [12]. Kamal et al. [16] worked on the feedback queueing model with four servers, one linked centrally with the other three servers having revisit atmost once but did not discussed the variations in the mean queue length with respect to the other parameters. For this reason, this article is being written. The present paper deals with the study of increase/ decrease of the mean queue length with respect to different queueing parameters.

Notation

λ: mean arrival rate at 1st server (S_1)

μ_1: mean service rate of 1st server (S_1)

μ_2: mean service rate of 2ndserver.

μ_3: mean service rate of 3rd server.

μ_4: mean service rate of 4th server

a_1**:** the probability of customer leaving1stserver 1sttime.

a_2: the probability of customer leaving1stserver 2nd time.

---

* Assistant Professor, Department of Mathematics, G.B.D.C. Rohtak, Haryana, India.
** Associate Professor, Department of Mathematics, Baba Mastnath University, Rohtak, Haryana, India.
*** Research Scholar, Department of Mathematics, Baba Mastnath University, Rohtak, Haryana, India.

b_1: the probability of customer leaving 2^{nd} server 1^{st} time.
b_2: the probability of customer leaving 2^{nd} server 2^{nd} time.
$C_1 = c_1$: the probability of customer leaving 3^{rd} server 1^{st} time.
$C_2 = c_2$: the probability of customer leaving 3^{rd} server 2^{nd} time.
d_1: the probability of customer leaving 4^{th} server 1^{st} time
d_2: the probability of customer leaving 4^{th} server 2^{nd} time
q_{12}: the probability of customer going from 1^{st} to 2^{nd} server 1^{st} time.
$q_{12'}$: the probability of customer going from 1^{st} to 2^{nd} server 2^{nd} time.
q_2: the probability of exit of customer from 2^{nd} server 1^{st} time.
q_{23}: the probability of customer going from 2^{nd} to 3^{rd} server 1^{st} time.
q_{21}: the probability of customer going from 2^{nd} to 1^{st} server 1^{st} time.
$q_{'2}$: the probability of exit of customer from 2^{nd} server 2^{nd} time.
$q_{23'}$: the probability of customer going from 2^{nd} to 3^{rd} server 2^{nd} time.
q'_{21}: the probability of customer going from 2^{nd} to 1^{st} server 2^{nd} time.
q_3: the probability of exit of customer from 3^{rd} server 1^{st} time.
q_{31}: the probability of customer going from 3^{rd} to 1^{st} server 1^{st} time.
q_{32}: the probability of customer going from 3^{rd} to 2^{nd} server 1^{st} time.
q_{34}: the probability of customer going from 3^{rd} to 4^{th} server 1^{st} time
$q_{3'}$: the probability of exit of customer from 3^{rd} server 2^{nd} time.
q'_{31}: the probability of customer going from 3^{rd} to 1^{st} server 2^{nd} time.
q'_{32}: the probability of customer going from 3^{rd} to 2^{nd} server 2^{nd} time.
q'_{34}: the probability of customer going from 3^{rd} to 4^{th} server 2^{nd} time.
q_4 : the probability of exit of customer from 4^{th} server 1^{st} time.
q'_4: the probability of exit of customer from 4^{th} server 2^{nd} time.
q_{41} : the probability of exit of customer from 4^{th} to 1^{st} server 1^{st} time.
q_{42}: the probability of customer going from 4^{th} to 2^{nd} server 1^{st} time.
q_{43}: the probability of customer going from 4^{th} to 3^{rd} server 1^{st} time.
q'_{42}: the probability of customer going from 4^{th} to 2^{nd} server 2^{nd} time.
q'_{43}: the probability of customer going from 4^{th} to 3^{rd} server 2^{nd} time.

Formulation of the Problem

The queue network consists of four service channels 1^{st}, 2^{nd}, 3^{rd} and 4^{th}; whereas 2^{nd}, 3^{rd} and 4^{th} are linked centrally with the 1^{st} server. It is assumed that customer arrive at 1^{st} server from outside the system according to a Poisson process with mean rate λ and then goes to 2^{nd}, 3^{rd} or 4^{th} servers for required services. After getting service at first server 1^{st} time, a customer either moves to the 2^{nd} ,3^{rd} or 4^{th} server such that $q_{12}+q_{13}+q_{14=1}$.

If the customer goes to second server after getting service 1^{st} time from the first server then, s/he either quits or moves back to the first server or to the third server or to the 4^{th} server such that $q_2+q_{21}+q_{23}+q_{24}=1$. Similarly, if the customer moves to third server from first server 1^{st} time then s/he either exits or revisits to the other servers and hence $q_3+q_{31}+q_{32}+q_{34}=1$.

From the fourth server, s/he may quit the system or move back to the lower order servers such that $q_4+q_{41}+q_{42}+q_{43}=1$. similarly, equations for revisit of the customer to any server can be written here. Let Q_{n_1,n_2,n_3,n_4} is the probability of having n_1, n_2, n_3, n_4 customers at server 1^{st}, 2^{nd}, 3^{rd} and 4^{th} at any time t.

The steady–state equations for different values of n_1, n_2, n_3, and n_4 are given by:

$$(\lambda + \mu_1 + \mu_2 + \mu_3 + \mu_4)\, Q_{n_1,n_2,n_3,n_4} = \lambda\, Q_{n_1-1,n_2,n_3,n_4} + \mu_1(a_1 q_{12} + a_2 q'_{12}) Q_{n_1+1,n_2-1,n_3,n_4}$$

$$+\mu_1(a_{14}q_{14} + a_2\, q'_{14})\, Q_{n_1+1,n_2,n_3,n_4-1} + \mu_1\,(a_1\, q_{13} + a_2\, q'_{13})\, Q_{n_1+1,n_2,n_3-1,n_4}$$

$$+ \mu_2\,(b_1\, q_{21})\, Q_{n_1-1 n_2+1,n_3,n_4} + \mu_2(b_1\, q_{21} + b_2 q'_2)\, Q_{n_1,n_2+1,n_3,n_4}$$

+ $\mu_2(b_1 q_{23} + b_2 q'_{23})Q_{n_1,n_2+1,n_3-1,n_4}$ + $\mu_2(b_1 q_{24} + b_2 q'_{24})\, Q_{n_1,n_2+1,n_3,n_4-1}$
+ $\mu_3 (c_1 q_3 + c_2 q'_3)\, Q_{n_1,n_2,n_3+1,}n_4$ + $\mu_3 c_1 q_{31}\, Q_{n_1-1,n_2,n_3+1},n_4$
+ $(c_1 q_{32} + c_2 q'_{32})\, \mu_3\, Q_{n_1,n_2-1,n_3+1,n_4}$ + $(c_1 q_{32} + c_2 q'_{32})\, \mu_3\, Q_{n_1,n_2-1,n_3+1,n_4}$
+ $(c_1 q_{34} + c_2 q'_{34})\, \mu_3\, Q_{n_1,n_2,n_3+1,n_4+1}$ + $(d_1 q_4 + d_2 q'_4)\mu_4 Q_{n_1,n_2,n_3,n_4+1}$
+ $(d_1 q_{41})\mu_4 Q_{n_1-1,n_2,n_3,n_4+1}$ + $(d_1 q_{42} + d_2 q'_{42})\mu_4 Q_{n_1,n_2-1,n_3,n_4+1}$
$+(d_1 q_{43} + d_2 q'_{43})\mu_4 Q_{n_1,n_2,n_3-1,n_4+1}$ …(1)

On solving this equation using generating function technique, we have the function:

$F(X,Y,Z,R) = f(X,Y,Z,R)/g(X,Y,Z,R)$ ………(2)

where

$$f(X,Y,Z,R) = \mu_1 F_0(Y,Z,R)\left[1 - \frac{1}{x}\{(a_1 q_{12} + a_2 q'_{12})y + (a_1 q_{13} + a_2 q'_{13})Z + (a_1 q_{14} + a_2 q'_{14})R\}\right]$$
$$+\mu_2 F_0(X,Z,R)\left[1 - \frac{1}{y}\{(b_1 q_2 + b_2 q'_2) + (b_1 q_{23} + b'_{23})Z + (b_1 q_{24} + b_2 q'_{24})R + (b_1 q_{21})x\}\right]$$
$$+ \mu_3 F_0(X,Y,R)\left[1 - \frac{1}{z}\{(C_1 q_3 + C_2 q'_3) + y(C_1 q_{32} + C_2 q'_{32}) + (C_1 q_{34} + C_2 q'_{34})R + (C_1 q_{31})x\}\right]$$
$$+ \mu_4 F_0(X,Y,Z)\left[1 - \frac{1}{R}\{(d_1 q_4 + d_2 q'_4) + y(d_1 q_{42} + d_2 q'_{42}) + Z(d_1 q_{43} + d_2 q'_{43}) + d_1 q_{41} x\}\right]$$

and

$$g(X,Y,Z,R) = \lambda(1-x) + \mu_1\left[1 - \frac{1}{x}\{(a_1 q_{12} + a_2 q'_{12})y + (a_1 q_{13} + a_2 q'_{13})Z + (a_1 q_{14} + a_2 q'_{14})R\}\right]$$
$$+\mu_2\left[1 - \frac{1}{y}\{(b_1 q_2 + b_2 q'_2) + (b_1 q_{23} + b'_{23})Z + (b_1 q_{24} + b_2 q'_{24})R + (b_1 q_{21})x\}\right]$$
$$+ \mu_3\left[1 - \frac{1}{Z}\{(C_1 q_3 + C_2 q'_3) + (C_1 q_{32} + C_2 q'_{32}) + (C_1 q_{34} + C_2 q'_{34})R + (C_1 q_{31})x\}\right]$$
$$+ \mu_4\left[1 - \frac{1}{R}\{(d_1 q_4 + d_2 q'_4) + (d_1 q_{42} + d_2 q'_{42})y + (d_1 q_{43} + d_2 q'_{43})Z + d_1 q_{41} x\}\right]$$

For convenience, let us define:

$F_1 = F_0(Y, Z, R)$
$F_2 = F_0(X, Z, R)$
$F_3 = F_0(X, Y, R)$
$F_4 = F_0(X, Y, Z)$

For x = y = Z = R = 1 and using the hypothesis/assumption reduces to in determinant form $\left(\frac{0}{0}\right)$.

Taking y = z = R = 1 and limit x→1 in (2) we have

$$-\lambda + \mu_1 - \mu_2 b_1 q_{21} - \mu_3 C_1 q_{31} - \mu_4 d_1 q_{41} = \mu_1 F_1 + b_1 q_{21}\mu_2 F_2 - C_1 q_{31}\mu_3 F_3 - d_1 q_{41}\mu_4 F_4 \quad (3)$$

Now, for x = z = R = 1 and taking y → 1 in (2):

(Using L'Hospital rule w.r.t y then but x = y = z = R =1)

$$-\mu_1(a_1 q_{12} + a_2 q'_{12}) + \mu_2 - \mu_3(C_1 q_{32} + C_2 q'_{32}) - \mu_4(d_1 q_{42} + d_2 q'_{42})$$
$$= -\mu_1 F_1(a_1 q_{12} + a_2 q'_{12}) + \mu_2 F_2 - \mu_3(C_1 q_{32} + C_2 q'_{32})F_3 - F_4\mu_4(d_1 q_{42} + d_2 q'_{42}) \quad (4)$$

For x = y = R = 1 and taking limit z → 1 in (2) we have:

$$-\mu_2(b_1 q_{23} + b_2 q'_{23}) - \mu_1(a_1 q_{13} + a_2 q'_{13}) + \mu_3$$
$$-\mu_4(d_1 q_{43} + d_2 q'_{43}) = -\mu_1 F_1(a_1 q_{13} + a_2 q'_{13}) - \mu_2 F_2(b_1 q_{23} + b_2 q'_{23})$$
$$+ \mu_3 F_3 - \mu_4 F_4(d_1 q_{43} + d_2 q'_{43}) \quad \ldots(5)$$

For x = y = Z = 1 and taking limit R → 1 in (2) we have :

$$-\mu_1(a_1 q_{14} + a_2 q'_{14}) - \mu_2(b_2 q_{24} + b_2 q'_{24}) - \mu_3(C_1 q_{34} + C_2 q'_{34}) + \mu_4$$
$$= -\mu_1(a_1 q_{14} + a_2 q'_{14})F_1 - \mu_2(b_1 q_{24} + b_2 q'_{24})F_2 - \mu_3(C_1 q_{34} + C_2 q'_{34})F_3 + \mu_4 F_4 \quad \ldots(6)$$

On solving (3), (4), (5) and (6), we have:

$$F_1=\frac{\begin{gathered}\mu_1 b_1(q_{21}(q_{12}(q'_{14}q_{23}-q'_{23})-q_{13}q'_{13}q_{23}-q'_{12}q'_{13}q'_{23})+(-q'_{12}q'_{14}-q_{13})q_{21}^2) +\lambda(q_{13}(\mu_4 q_{23}+q'_{21}) \\ +q'_{14}q_{23}+q'_{12}(q'_{23}\mu_4+q'_{14}q'_{21})-q'_{23})+\mu_1(q_{13}(-\mu_4 q_{23}-q'_{21})-q'_{14}q_{23} +q'_{12}(-q'_{23}\mu_4-q'_{14}q'_{21}) \\ +q'_{23})+q_{41}\mu_1 d_1(q_{12}(-\mu_4 q_{23}-q'_{21})-q'_{13}q_{23}+q_{21}(q'_{12}\mu_4-1)-q'_{12}q'_{13}q'_{21}) +q_{31}\mu_1(q_{12} \\ (-q'_{23}\mu_4-q'_{14}q'_{21})+q_{21}(-q_{13}\mu_4-q'_{14})-q'_{13}q'_{23}+q_{13}q'_{13}q'_{21})C_1\end{gathered}}{\begin{gathered}\mu_1 b_1(q_{21}(q_{12}(q'_{14}q_{23}-{q'}_{23})-q_{13}q'_{13}q_{23}-q'_{12}q'_{13}q'_{23})+(-q'_{12}q'_{14}-q_{13})q_{21}^2) +\mu_1(q_{13}(-\mu_4 q_{23} \\ -q'_{21})-q'_{14}q_{23}+q'_{12}(-q'_{23}\mu_4-q'_{14}q'_{21})+q'_{23}) +q_{41}\mu_1 d_1(q_{12}(-\mu_4 q_{23}-q'_{21})-q'_{13}q_{23} \\ +q_{21}(q'_{12}\mu_4-1)-q'_{12}q'_{13}q'_{21})+q_{31}\mu_1(q_{12}(-q'_{23}\mu_4-q'_{14}q'_{21}) +q_{21}(-q_{13}\mu_4-q'_{14}) \\ -q'_{13}q'_{23}+q_{13}q'_{13}q'_{21})C_1\end{gathered}}$$

$$F_2=\frac{\begin{gathered}\mu_2 b_1(q_{21}(q_{12}(q'_{14}q_{23}-q'_{23})-q_{13}q'_{13}q_{23}-q'_{12}q'_{13}q'_{23})+(-q'_{12}q'_{14}-q_{13})q_{21}^2) \\ +\mu_2(q_{13}(-\mu_4 q_{23}-q'_{21})-q'_{14}q_{23}+q'_{12}(-q'_{23}\mu_4-q'_{14}q'_{21})+q'_{23})+q'_{23})+q_{41}\mu_2 \\ d_1(q_{12}(-\mu_4 q_{23}-q'_{21})-q'_{13}q_{23}+q_{21}(q'_{12}\mu_4-1)-q'_{12}q'_{13}q'_{21})+\lambda(q_{12}(q'_{14}q_{23}-q'_{23}) \\ -q_{13}q'_{13}q_{23}-q'_{12}q'_{13}q'_{23},+(-q'_{12}q'_{14}-q_{13})q_{21})+q_{31}\mu_{2(q_{12}}(-q'_{23}\mu_4-q'_{14}q'_{21}) \\ +q_{21}(-q_{13}\mu_4-q'_{14})-q'_{13}q'_{23}+q_{13}q'_{13}q'_{21})C_1\end{gathered}}{\begin{gathered}\mu_2 b_1(q_{21}(q_{12}(q'_{14}q_{23}-q'_{23})-q_{13}q'_{13}q_{23}-q'_{12}q'_{13}q'_{23})+(-q'_{12}q'_{14}-q_{13})q_{21}^2)+ \\ \mu_2(q_{13}(-\mu_4 q_{23}-q'_{21})-q'_{14}q_{23}+q'_{12}(-q'_{23}\mu_4-q'_{14}q'_{21})+q'_{23})+q_{41}\mu_2 d_1 \\ (q_{12}(-\mu_4 q_{23}-q'_{21})-q'_{13}q_{23}+q_{21}(q'_{12}\mu_4-1)-q'_{12}q'_{13}q'_{21})+q_{31}\mu_2 \\ (q_{12}(-q'_{23}\mu_4-q'_{14}q'_{21})+q_{21}(-q_{13}\mu_4-q'_{14})-q'_{13}q'_{23}+q_{13}q'_{13}q'_{21})C_1\end{gathered}}$$

$$F_3=\frac{\begin{gathered}\mu_3 b_1(q_{21}(q_{12}(q'_{14}q_{23}-q'_{23})-q_{13}q'_{13}q_{23}-q'_{12}q'_{13}q'_{23})+(-q'_{12}q'_{14}-q_{13})q_{21}^2) \\ +\mu_3(q_{13}(-\mu_4 q_{23}-q'_{21})-q'_{14}q_{23}+q'_{12}(-q'_{23}\mu_4-q'_{14}q'_{21})+q'_{23})+q_{41}\mu_3 d_1 \\ (q_{12}(-\mu_4 q_{23}-q'_{21})-q'_{13}q_{23}+q_{21}(q'_{12}\mu_4-1)-q'_{12}q'_{13}q'_{21})+q_{31}\mu_3(q_{12} \\ (-q'_{23}\mu_4-q'_{14}q'_{21})+q_{21}(-q_{13}\mu_4-q'_{14})-q'_{13}q'_{23}+q_{13}q'_{13}q'_{14})C_1 \\ +\lambda(q_{12}(-q'_{23}\mu_4-q'_{14}q'_{21})+q_{21}(-q_{13}\mu_4-q'_{14})-q'_{13}q'_{23}+q_{13}q'_{13}q'_{21})\end{gathered}}{\begin{gathered}\mu_3 b_1(q_{21}(q_{12}(q'_{14}q_{23}-q'_{23})-q_{13}q'_{13}q_{23}-q'_{12}q'_{13}q'_{23})+(-q'_{12}q'_{14}-q_{13})q_{21}^2) \\ +\mu_3(q_{13}(-\mu_4 q_{23}-q'_{21})-q'_{14}q_{23}+q'_{12}(-q'_{23}\mu_4-q'_{14}q'_{21})+q'_{23})+q_{14}\mu_3 d_1 \\ (q_{12}(-\mu_4 q_{23}-q'_{21})-q'_{13}q_{23}+q_{21}(q'_{12}\mu_4-1)-q'_{12}q'_{13}q'_{21})+q_{31}\mu_3(q_{12} \\ (-q'_{23}\mu_4-q'_{14}q'_{21})+q_{21}(-q_{13}\mu_4-q'_{14})-q'_{13}q'_{23}+q_{13}q'_{13}q'_{21})C_1\end{gathered}}$$

$$F_4=1+\frac{\lambda(q_{12}(-\mu_4 q_{23}-q'_{21})-q'_{13}q_{23}+q_{21}(q'_{12}\mu_4-1)-q'_{12}q'_{13}q'_{21})}{\begin{gathered}\mu_4 b_1(q_{21}(q_{12}(q'_{14}q_{23}-q'_{23})-q_{13}q'_{13}q_{23}-q'_{12}q'_{13}q'_{23})+(-q'_{12}q'_{14}-q_{13})q_{21}^2) \\ +\mu_4(q_{13}(-\mu_4 q_{23}-q'_{21})-q'_{14}q_{23}+q'_{12}(-q'_{23}\mu_4-q'_{14}q'_{21})+q'_{23})+q_{14}\mu_4 d_1(q_{12}(-\mu_4 q_{23} \\ -q'_{21})-q'_{13}q_{23}+q_{21}(q'_{12}\mu_4-1)-q'_{12}q'_{13}q'_{21})+q_{31}\mu_4(q_{12}(-q'_{23}\mu_4-q'_{14}q'_{21}) \\ +q_{21}(-q_{13}\mu_4-q'_{14})-q'_{13}q'_{23}+q_{13}q'_{13}q'_{21})C_1\end{gathered}}$$

Let Lq_1 , Lq_2 , Lq_3 and Lq_4 denote the mean queue length at 1st , 2nd, 3rd and 4th server then we have:

$$Lq_1=\frac{\left(\frac{\partial f}{\partial X}\right)_{(1,1,1,1)}\left(\frac{\partial^2 g}{\partial X^2}\right)_{(1,1,1,1)}+\left(\frac{\partial g}{\partial X}\right)_{(1,1,1,1)}\left(\frac{\partial^2 f}{\partial X^2}\right)_{(1,1,1,1)}}{2\left[\left(\frac{\partial g}{\partial X}\right)_{(1,1,1,1)}\right]^2}$$

$$Lq_1=\frac{\begin{gathered}(\mu_1F_1-b_1q_{21}\mu_2F_2-c_1q_{31}\mu_3F_3-d_1q_{41}\mu_4F_4)(-2\mu_1)+ \\ (-\lambda+\mu_1-b_1q_{21}\mu_2-c_1q_{31}\mu_3-d_1q_{41}\mu_4)(-2\mu_1F_1)\end{gathered}}{\begin{gathered}2[-\lambda+\mu_1\{(a_1q_{12}+a_2q'_{12})+(a_1q_{13}+a_2q'_{13})+(a_1q_{14}+a_2q'_{14})\}-\mu_2b_1q_{21} \\ -\mu_3c_1q_{31}-\mu_4d_1q_{41}]^2\end{gathered}}$$

$$Lq_2=\frac{\left(\frac{\partial f}{\partial Y}\right)_{(1,1,1,1)}\left(\frac{\partial^2 g}{\partial Y^2}\right)_{(1,1,1,1)}+\left(\frac{\partial g}{\partial Y}\right)_{(1,1,1,1)}\left(\frac{\partial^2 f}{\partial Y^2}\right)_{(1,1,1,1)}}{2\left[\left(\frac{\partial g}{\partial Y}\right)_{(1,1,1,1)}\right]^2}$$

$$=\frac{\begin{gathered}[-\mu_1F_1(a_1q_2+a_2q'_{12})+\mu_2F_2-\mu_3F_3(c_1q_{32}+c_2q'_{32})-\mu_4F_4(d_1q_{42}+d_2q'_{42})](-2\mu_2)+ \\ [-\mu_1(a_1q_{12}+a_2q'_{12})+\mu_2-\mu_3(c_1q_{32}+c_2q'_{32})-\mu_4(d_1q_{42}+d_2q'_{42})](2-\mu_2F_2)\end{gathered}}{2[-\mu_1(a_1q_{12}+a_2q'_{12})+\mu_2-\mu_3(c_1q_{32}+c_2q'_{32})-\mu_4(d_1q_{42}+d_2q'_{42})]^2}$$

$$Lq_3 = \frac{\left(\frac{\partial f}{\partial Z}\right)_{(1,1,1,1)}\left(\frac{\partial^2 g}{\partial Z^2}\right)_{(1,1,1,1)} + \left(\frac{\partial g}{\partial Z}\right)_{(1,1,1,1)}\left(\frac{\partial^2 f}{\partial Z^2}\right)_{(1,1,1,1)}}{2\left[\left(\frac{\partial g}{\partial Z}\right)_{(1,1,1,1)}\right]^2} Lq_3$$

$$= \frac{\begin{array}{c}[-\mu_1F_1(a_1q_{13}+a_2q'_{13})-\mu_2F_2(b_1q_{23}+b_2q'_{23})+\mu_3F_3-\mu_4F_4(d_1q_{43}+d_2q'_{43})](-2\mu_3)+\\ [-\mu_1(a_1q_{13}+a_2q'_{13})-\mu_2(b_1q_{23}+b_2q'_{23})+\mu_3-\mu_4(d_1q_{43}+d_2q'_{43})](-2\mu_3F_3)\end{array}}{2[-\mu_1(a_1q_{13}+a_2q'_{13})-\mu_2(b_1q_{23}+b_2q'_{23})+\mu_3-\mu_4(d_1q_{43}+d_2q'_{43})]^2} Lq_4$$

$$= \frac{\left(\frac{\partial f}{\partial R}\right)_{(1,1,1,1)}\left(\frac{\partial^2 g}{\partial R^2}\right)_{(1,1,1,1)} + \left(\frac{\partial g}{\partial R}\right)_{(1,1,1,1)}\left(\frac{\partial^2 f}{\partial R^2}\right)_{(1,1,1,1)}}{2\left[\left(\frac{\partial g}{\partial R}\right)_{(1,1,1,1)}\right]^2}$$

$$Lq_4 = \frac{\begin{array}{c}[-\mu_1F_1(a_1q_{14}+a_2q'_{14})-\mu_2F_2(b_1q_{24}+b_2q'_{24})-\mu_3F_3(c_1q_{34}+c_2q'_{34})+\mu_4F_4](-2\mu_4)+\\ [-\mu_1(a_1q_{14}+a_2q'_{14})-\mu_2(b_1q_{24}+b_2q'_{24})-\mu_3(c_1q_{34}+c_2q'_{34})+\mu_4(-2\mu_4F_4)\end{array}}{2[-(a_1q_{14}+a_2q'_{14})\mu_1-(b_1q_{24}+b_2q'_{24})\mu_2-(c_1q_{34}+c_2q'_{34})\mu_3+\mu_4]^2}$$

Let Lq denote the mean queue length of the queueing system. Then we have:

$Lq = Lq_1 + Lq_2 + Lq_3 + Lq_4$

$$Lq = \frac{\begin{array}{c}(\mu_1F_1-b_1q_{21}\mu_2F_2-c_1q_{31}\mu_3F_3-d_1q_{41}\mu_4F_4)(-2\mu_1)+\\ (-\lambda+\mu_1-b_1q_{21}\mu_2-c_1q_{31}\mu_3-d_1q_{41}\mu_4)(-2\mu_1F_1)\end{array}}{2[-\lambda+\mu_1\{(a_1q_{12}+a_2q'_{12})+(a_1q_{13}+a_2q'_{13})+(a_1q_{14}+a_2q'_{14})\}-\mu_2b_1q_{21}-\mu_3c_1q_{31}-\mu_4d_1q_{41}]^2} +$$

$$\frac{\begin{array}{c}[-\mu_1F_1(a_1q_2+a_2q'_{12})+\mu_2F_2-\mu_3F_3(c_1q_{32}+c_2q'_{32})-\mu_4F_4(d_1q_{42}+d_2q'_{42})](-2\mu_2)+\\ [-\mu_1(a_1q_{12}+a_2q'_{12})+\mu_2-\mu_3(c_1q_{32}+c_2q'_{32})-\mu_4(d_1q_{42}+d_2q'_{42})](2-\mu_2F_2)\end{array}}{2[-\mu_1(a_1q_{12}+a_2q'_{12})+\mu_2-\mu_3(c_1q_{32}+c_2q'_{32})-\mu_4(d_1q_{42}+d_2q'_{42})]^2}$$

$$+ \frac{\begin{array}{c}[-\mu_1F_1(a_1q_{13}+a_2q'_{13})-\mu_2F_2(b_1q_{23}+b_2q'_{23})+\mu_3F_3-\mu_4F_4(d_1q_{43}+d_2q'_{43})](-2\mu_3)+\\ [-\mu_1(a_1q_{13}+a_2q'_{13})-\mu_2(b_1q_{23}+b_2q'_{23})+\mu_3-\mu_4(d_1q_{43}+d_2q'_{43})](-2\mu_3F_3)\end{array}}{2[-\mu_1(a_1q_{13}+a_2q'_{13})-\mu_2(b_1q_{23}+b_2q'_{23})+\mu_3-\mu_4(d_1q_{43}+d_2q'_{43})]^2} +$$

$$\frac{\begin{array}{c}[-\mu_1F_1(a_1q_{14}+a_2q'_{14})-\mu_2F_2(b_1q_{24}+b_2q'_{24})-\mu_3F_3(c_1q_{34}+c_2q'_{34})+\mu_4F_4](-2\mu_4)+\\ [-\mu_1(a_1q_{14}+a_2q'_{14})-\mu_2(b_1q_{24}+b_2q'_{24})-\mu_3(c_1q_{34}+c_2q'_{34})+\mu_4(-2\mu_4F_4)\end{array}}{2[-(a_1q_{14}+a_2q'_{14})\mu_1-(b_1q_{24}+b_2q'_{24})\mu_2-(c_1q_{34}+c_2q'_{34})\mu_3+\mu_4]^2}$$

Numerical Results and Discussion

- **Behaviour of Mean Queue Length (Lq) of the system with respect to λ (the mean arrival rate) for different values of a_1 (the probability of leaving 1st server 1st time) is depicted in Table 1 and in Fig.1 keeping the values of other parameters as fixed.**

Table 1

$\mu_1=1$, $\mu_2=3$, $\mu_3=5$, $\mu_4=0.2$, $b_1=0.7$, $b_2=0.3$, $c_1=0.5$, $c_2=0.5$, $d_1=0.8$, $d_2=0.2$, $q_{13}=0.3$, $q_{12}=0.2$, $q_{14}=0.5$, $q_{13}'=0.2$, $q_{12}'=0.1$, $q_{14}'=0.7$ $q_2=0.1$, $q_{21}=0.4$, $q_{23}=0.3$, $q_{24}=0.2$, $q'_2=0.3$, $q'_{21}=0.3$, $q'_{23}=0.1$, $q'_{24}=0.3$, $q_3=0.6$, $q_{31}=0.2$, $q_{32}=0.15$, $q_{34}=0.05$, $q_3'=0.2$, $q_{32}'=0.3$, $q'_{34}=0.5$, $q_4=0.7$, $q_{41}=0.1$, $q_{42}=0.15$, $q_{43}=0.05$, $q'_4=0.8$, $q'_{42}=0.15$, $q'_{43}=0.05$			
λ	**a_1=0.6**	**a_1=0.7**	**a_1=0.8**
1	22.22261	21.56559	20.91802
2	39.27666	38.12311	36.98634
3	52.71691	51.06685	49.44086
4	62.54338	60.3968	58.2816
5	68.75606	66.11295	63.50855
6	71.35494	68.21532	65.1217
7	70.34004	66.7039	63.12107
8	65.71135	61.57868	57.50665
9	57.46887	52.83968	48.27844

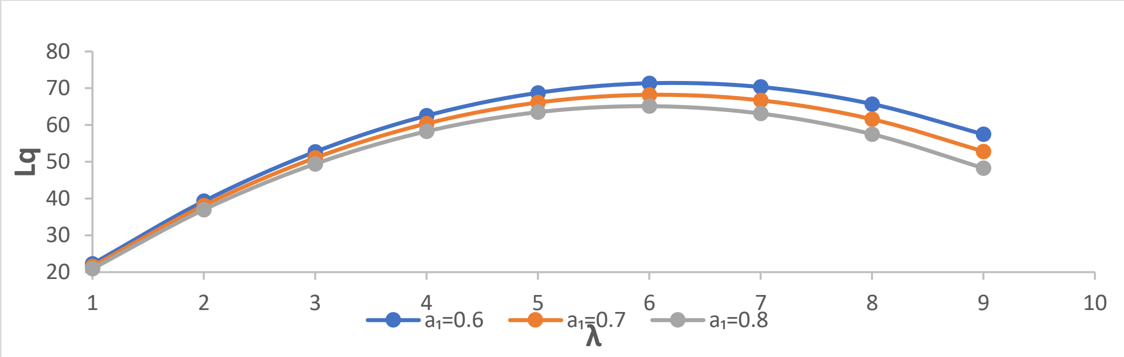

Fig 1

Following can be interpreted from **Table 1** and **Fig. 1**:

- Mean queue length Lq of the system increases for $\lambda \leq 6$ and decreases for $\lambda > 6$ with the increase in the probability a_1.
- Mean queue length Lq of the system decreases with increase in with the increase in a_1.

- **Behaviour of Mean Queue Length (Lq) of the system with respect to μ_1 (the mean service rate of the 1st server) for different values of μ_2 (the mean service rate of the 2nd server) is depicted in Table 2 and in Fig.2 keeping the values of other parameters as fixed.**

Table 2

$\lambda=1, a_1=0.6, a_2=0.4, \mu_3=5, \mu_4=0.2, b_1=0.7, b_2=0.3, c_1=0.5, c_2=0.5, d_1=0.8, d_2=0.2, q_{13}=0.3, q_{12}=0.2, q_{14}=0.5, q_{13}'=0.2, q_{12}'=0.1, q_{14}'=0.7$ $q_2=0.1$, $q_{21}=0.4, q_{23}=0.3, q_{24}=0.2, q'_2=0.3, q'_{21}=0.3, q'_{23}=0.1, q'_{24}=0.3, q_3=0.6, q_{31}=0.2, q_{32}=0.15, q_{34}=0.05, q_3'=0.2, q_{32}'=0.3, q'_{34}=0.5, q_4=0.7, q_{41}=0.1, q_{42}=0.15, q_{43}=0.05, q'_4=0.8, q'_{42}=0.15, q'_{43}=0.05$			
μ_1	**$\mu_2=3$**	**$\mu_2=4$**	**$\mu_2=5$**
1	22.22261	31.26786	41.19353
2	43.72358	58.29173	73.86175
3	66.66512	87.44017	109.2694
4	94.65527	122.4997	151.3747
5	130.1196	165.9546	202.6914
6	175.2683	220.0635	265.4773
7	232.2263	287.0139	341.8998
8	303.0525	368.9483	434.0549
9	389.8128	467.9531	543.9272

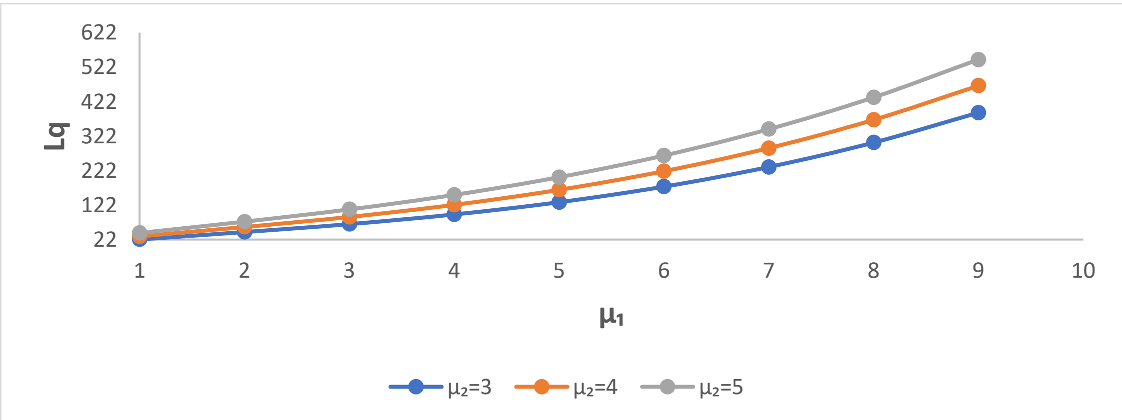

Fig. 2

Following can be interpreted from **Table 2** and **Fig. 2**:

- Mean queue length Lq of the system increases with respect to μ_1.
- Mean queue length Lq of the system increases with respect to μ_2.

References

4. Atar R and Mendelson G, "On the non-markovian multi-class queue under risk-sensitive cost", Queuing Systems, 84 (3-4), 2016, 265-78.
5. Atencia, I and Moreno, P; "Discrete-time Geo$^{[X]}$/G_H/1 Retrial Queue with Bernoulli Feedback", Computers & Mathematics with Applications, 47, (8–9) , April–May 2004, 1273-94.
6. Avrachenkov K, Morozov E and Steyaert B, "Sufficient stability conditions for multi class constant retrial rate systems", Queuing Systems, 82(1-2), 2016, 149-71.
7. Ginting D.A., Aristotles and Endah O.D., "Implementation of multi-level feedback queue algorithm in restaurant order food application development for android and ios platforms", International Journals of Computer Applications, 180 (13), 2013, 24-30.
8. Gupta, V.K., Joshi T.N. and Tiwari, S.K., "A balked M/D/1 feedback queuing models", IOSR- JM, 12(1), 2016, 83-87.
9. Kalyanaraman R and Suvitha V.; "A single server vacation queue with type of services and with restricted admissibility", Far East Journal of Mathematical Sciences, 103 (1) 2018, 125-157.
10. Ke J.C. and Chang F.M., "Modified vacation policy for M/G/1 retrial queue with balking and feedback", Computers and Industrial Engineering, 57 (1), 2009, 433-43.
11. Kumar B.K., Arivudainambi D and Krishanmoorthy A, "Some results on a generalized M/G/1 feedback queue with negative customers", Annals of Operation Research, 143(1), 2006, 277-96.
12. Kumar, B.K, Madheshwari S.P. and Kumar, A.V., "The M/G/1 retrial queue with feedback and starting failures", Applied Mathematical Modeling, 26 (11), 2002, 1057-75.
13. Kumar, B.K, Madheshwari, S.P. and Lakshmi S.R.A., "An M/G/1 Bernoulli feedback retrial queuing system with negative customers", Operation Research, 13(2), 2013, 187-210.
14. Kumar B.K and Raja J, "On multi-server feedback retrial queues with balking and control retrial rate", Annals of Operation Research, 141(1), 2006, 211-32.
15. Kumar S. and Taneja G.; "Analysis of a hierarchical structured queuing system with three service channels, and chances of repetition of service with feedback", International Jounal of Pure and Applied Mathematics, 115 (2) 2017, 383-363.
16. Latouche, Guy, Nguyen and Giang T.; "Feedback Control: Two-Sided Markov-Modulated Brownian Motion with Instantaneous Change of Phase at Boundaries", Performance Evaluation, 106, December 2016, 30–49.
17. Sundri S.M. and Srinivasan S, "Multi-phase M/G/1 queue with Bernoulli feedback and multiple server vacation", 52(1), 2012, 18-23.
18. Tang J and Zhao Y.Q., "Stationary tail asymptotic of a tandem queue with feedback", Annals of Operation Research, 160 (1), 2008,173-89.
19. Kamal et al. (2023). Mathematical modeling of some feedback queueing systems. Ph. D. Thesis, Baba Mastnath University, Asthal Bohar, Rohtak, Haryana..

□❍□