International Journal of Innovations & Research Analysis (IJIRA)
(An International Quarterly Multidisciplinary Peer Reviewed Refereed Journal)
ISSN: 2583-0295 (Online), Impact Factor: (Cosmos 7.662 & I2OR 5.105), Volume 06, No. 03(I), July-September, 2026, pp 153-169

Quantum Calculus Meets Toeplitz: Sharp Bounds for p-Valent Ma-Minda Classes

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Citation: Kumar, P. & Soni, A. (2026). Quantum Calculus Meets Toeplitz: Sharp Bounds for p-Valent Ma-Minda Classes. International Journal of Innovations & Research Analysis, 06(03(I)), 153–169. https://doi.org/10.62823/IJIRA/06.3(I).9344

Abstract

In this paper, we investigate Toeplitz determinants associated with subclasses of p-valent q-starlike and q-convex Ma-Minda functions defined via the Jackson q-derivative operator. By employing subordination techniques together with coefficient estimates for functions having positive real part, we derive upper bounds for the Toeplitz determinants $T_2(p+1)$ and $T_3(p)$. In several cases, the obtained estimates are shown to be sharp through appropriate extremal functions. Our results unify and extend various earlier results available for classical univalent and multivalent function classes. Furthermore, by letting $q \to 1^-$, the corresponding results for the classical Ma-Minda starlike and convex classes are recovered as special cases. Applications to certain subclasses associated with Janowski, exponential, and cardioid-type domains are also discussed.

***Keywords**: Multivalent Functions, q-Starlike Function, q-Convex Functions, Subordination, Toeplitz Determinants, and Coefficient Bounds.*

Introduction

Let $\mathcal{A}_p$ denote the class of the functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, (p \in \mathbb{N}, z \in \mathcal{U}). \tag{1.1}$$

In particular, for $p = 1$, we denote $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{A}$ is said to be univalent in $\mathcal{U}$, denoted as $f \in \mathcal{S}$, if it satisfies $f(z_1) \neq f(z_2)$ for any distinct $z_1, z_2 \in \mathcal{U}$. The notion of subordination for functions in $\mathcal{A}$ is introduced as follows: A function f is subordinate to g in $\mathcal{U}$, denoted by $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z)$ satisfying specific conditions such that $f(z) = g(\omega(z))$ for all $z \in \mathcal{U}$. Furthermore, if g is univalent in $\mathcal{U}$, then f is subordinate to g if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$. Additional details on the Schwarz function can be found in [15]. Let $\mathcal{P}$ denote the class of analytic functions p defined in the open unit disk $\mathcal{U}$, such that $p(0) = 1$ and $\Re(p(z)) > 0$. Furthermore, $p \in \mathcal{P}$ if and only if the subordination $p(z) \prec (1+z)/(1-z)$ holds. It is a recognized principle that a function $f \in \mathcal{S}$ is categorized as starlike $(f \in \mathcal{S}^*)$ or convex $(f \in \mathcal{C})$ provided there exists a function $p \in \mathcal{P}$ for which $zf'(z)/f(z) = p(z)$ or $1 + zf''(z)/f'(z) = p(z)$ is satisfied for all $z \in \mathcal{U}$, respectively. For a more comprehensive discussion on these function classes and their associated properties, one may refer to [1] and [15].

Let $\psi(z)$ be an analytic function in $\mathcal{U}$ with a positive real part, satisfying $\psi(0) = 1$ and $\psi'(0) > 0$ and mapping $\mathcal{U}$ onto a region that is starlike with respect to 1 and symmetric about the real axis. In such cases, $\psi(z)$ admits an expansion of the form

$$\psi(z) = 1 + B_1 z + B_2 z^2 + \cdots, (B_1 > 0, z \in \mathcal{U}).$$

The class $\mathcal{S}_p^*(\psi)$ consists of p-valent starlike functions classes for which $f \in \mathcal{A}_p$ and satisfying the subordination condition

$$\frac{1}{p}\frac{zf'(z)}{f(z)} \prec \psi(z), (z \in \mathcal{U}), \tag{1.2}$$

while the class $\mathcal{C}_p(\psi)$ of p-valent convex function class and is defined by the condition

$$\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \psi(z), (z \in \mathcal{U}). \tag{1.3}$$

Ali et al. [5] studied these classes and established sharp distortion, growth, covering and rotation theorems. The classes $\mathcal{S}_p^*(\psi)$ and $\mathcal{C}_p(\psi)$ include several wellknown subclasses of p-valent starlike and convex functions as special cases.

For an analytic function of the form

$$\psi(z) = \frac{1+\left(1-\frac{2\gamma}{p}\right)z}{1-z},$$

the definition of $\mathcal{S}_p^*(\psi)$ leads to

$$\frac{zf'(z)}{f(z)} \prec \frac{p+(p-2\gamma)z}{1-z}, (0 \le \gamma < p, z \in \mathcal{U}).$$

A function satisfying the above condition belongs to the class of p-valent starlike functions of order γ, denoted by $\mathcal{S}_p^*(\gamma)$. For $p = 1$, the classes $\mathcal{S}_p^*(\psi)$ and $\mathcal{C}_p(\psi)$ were introduced and analyzed by Ma and Minda (see [29]). Additionally, we define

$$\mathcal{S}_p^*(0) = \mathcal{S}_p^*, \mathcal{S}_1^*(\gamma) = \mathcal{S}^*(\gamma) \text{ and } \mathcal{S}_1^*(0) = \mathcal{S}^*.$$

Quantum calculus, commonly referred to as q-calculus, is a branch of mathematics that operates independently of limits. Its significance extends across various fields of mathematics and physics, where it plays a crucial role in areas such as ordinary and fractional calculus, basic hypergeometric functions, orthogonal polynomials, and combinatorics. The versatility of q-calculus makes it a powerful tool for addressing complex mathematical problems across multiple disciplines. Let $q \in (0,1)$, the q derivative (also known as the q-difference operator), originally introduced by Jackson [20], is defined as follows,

$$(\mathcal{D}_q f)(z) = \frac{f(z)-f(qz)}{(1-q)z}, (z \neq 0).$$

It is important to note that if f is differentiable at z, then

$$\lim_{q\to 1^-}(\mathcal{D}_q f)(z) = f'(z).$$

For a function f of the form (1.1), the q-derivative takes the form

$$(\mathcal{D}_q f)(z) = 1 + \sum_{n=2}^{\infty}[n]_q a_n z^{n-1}$$

where

$$[n]_q = \frac{1-q^n}{1-q}, q \in (0,1).$$

Clearly, as $q \to 1^-$, we observe that $[n]_q \to n$, recovering the standard derivative. Also, we have

$$[n+p]_q = [n]_q + q^n[p]_q = q^p[n]_q + [p]_q$$
$$[n-p]_q = q^{-p}[n]_q - q^{-p}[p]_q$$
$$[0]_q = 0, [1]_q = 1$$

For further details on the definitions and properties of q-calculus, the reader may refer to [11], [20] and [23]. In this article we now use the following multivalent classes of starlike and convex function for which the various Toeplitz bounds are obtained in the light of q-calculus.

- **Definition 1.** The class $\mathcal{S}^*_{p,q}$ of q-starlike functions was proposed and examined by Ismail et al. in[19]. A function $f \in \mathcal{A}_p$ is said to belong to the class $\mathcal{S}^*_{p,q}$ if and only if
$$\mathcal{S}^*_{p,q}(\varphi) = \left\{ f \in \mathcal{A}_p : \frac{1}{[p]_q}\left(\frac{z\mathcal{D}_q f(z)}{f(z)}\right) \prec \varphi(z), q \in (0,1), z \in \mathcal{U} \right\}.$$
In the limiting case $q \to 1^-$together with $p = 1$, the class $\mathcal{S}^*_{p,q}(\varphi)$ reduces to the class $\mathcal{S}^*(\varphi)$.
- **Definition 2.** The class $\mathcal{C}_{p,q}$ of q-convex functions was introduced by Ahuja et al. in [2]. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{C}_{p,q}$ if and only if
$$\mathcal{C}_{p,q}(\varphi) = \left\{ f \in \mathcal{A}_p : \frac{1}{[p]_q}\left(\frac{\mathcal{D}_q\big(z\mathcal{D}_q f(z)\big)}{\mathcal{D}_q f(z)}\right) \prec \varphi(z), q \in (0,1), z \in \mathcal{U} \right\}.$$
Further, in the limiting case $q \to 1^-$and $p = 1$, the class $\mathcal{C}_{p,q}(\varphi)$ reduces to the class $\mathcal{C}(\varphi)$.

Hankel and Toeplitz matrices share a close relationship. A Hankel matrix is defined by the property that its entries remain constant along each reverse diagonal. Ye and Lim [35] showed that, in general, any $n \times n$ matrix over $\mathbb{C}$ can be expressed as a product of certain Toeplitz or Hankel matrices. More recently, S. Giri and S. Sivaprasad Kumar [13] introduced the notion of a Toeplitz determinant whose entries are formed from the logarithmic coefficients of functions belonging to the class $\mathcal{A}$. They investigated the precise bounds $T_{2,1}(f) = \gamma_1^2 - \gamma_2^2$ and $T_{2,2}(f) = \gamma_2^2 - \gamma_3^2$ for the classes $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$. One can see that B. Bhowmik et al. [8] obtained the successive coefficients and toeplitz determinants for concave univalent functions class $Co(p)$ defined in the unit disc $\mathcal{U}$ having a simple pole at $z = p$. Similarly in 2023, V. Arora et al. [7] determine the successive coefficients for functions in the spirallike family. Actually they compute the bounds of successive coefficients $\big||a_{n+1}| - |a_n|\big|$ for the family of univalent functions that are spirallike functions of non-negative order. These results provide the alternate simple proof for the case of univalent spirallike functions.\

In 2020, A. Lecko [28] determined the fourth-order Hermitian Toeplitz determinant for convex functions. The same author further established sharp bounds for the Hermitian Toeplitz determinants corresponding to several subclasses of close-toconvex functions; see [28]. Moreover, V. Kumar and S. Kumar [26] investigated bounds for Hermitian Toeplitz and Hankel determinants associated with strongly starlike functions. More recently, T. G. Shaba [17] derived boundary values of Hankel and Toeplitz determinants for the class of q convex functions by introducing a new q-differential operator defined via generalized binomial series. In a similar direction, P. Jastrzębski et al. [22] obtained the Hermitian Toeplitz determinants of second and third order for certain subclasses of close-to-star functions. It is important to note that Toeplitz symmetric matrices are characterized by constant entries along each diagonal. For a given function
$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n$$
we associate a determinant $T_m(n)$ defined as follows:
$$T_m(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_n & \cdots & a_{n+m} \\ \vdots & \vdots & & \vdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_n \end{vmatrix},$$
where $m, n \in \mathbb{N}$. The bounds of $|T_2(n)|, |T_3(1)|$, and $|T_3(2)|$ for the class $\mathcal{S}^*$ and $\mathcal{S}$ were investigated by Ali et al.[6] in 2017. After that, for small values of m and n, several authors studied the Toeplitz coefficient bounds $T_m(n)$ for various subclasses of $\mathcal{S}$ in recent years. Especially [[9],[10],[12],[25]] obtained the coefficient bounds of various subclasses of starlike and convex functions. Inspired by these studies and recognizing the importance of the Toeplitz determinant for univalent classes, we now introduce the p-valent Toeplitz determinant associated with function
$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$$

as follows:

$$T_m(n+p)=\begin{vmatrix} a_{n+p} & a_{n+p+1} & \cdots & a_{n+p+m-1}\\ a_{n+p+1} & a_{n+p} & \cdots & a_{n+p+m}\\ \vdots & \vdots & & \vdots\\ a_{n+p+m-1} & a_{n+p+m} & \cdots & a_{n+p}\end{vmatrix}.$$

Consequently, we obtain

$$T_2(p+1)=\begin{vmatrix} a_{p+1} & a_{p+2}\\ a_{p+2} & a_{p+1}\end{vmatrix}, T_2(p+2)=\begin{vmatrix} a_{p+2} & a_{p+3}\\ a_{p+3} & a_{p+2}\end{vmatrix},$$

$$T_3(p)=\begin{vmatrix} 1 & a_{p+1} & a_{p+2}\\ a_{p+1} & 1 & a_{p+1}\\ a_{p+2} & a_{p+1} & 1\end{vmatrix} \text{ and } T_3(p+1)=\begin{vmatrix} a_{p+1} & a_{p+2} & a_{p+3}\\ a_{p+2} & a_{p+1} & a_{p+2}\\ a_{p+3} & a_{p+2} & a_{p+1}\end{vmatrix} \text{ etc.}$$

Also we note that

$$T_2(p+1)=a_{p+1}^2-a_{p+2}^2$$

and

$$T_3(p)=1-2a_{p+1}^2-a_{p+2}\left(a_{p+2}-2a_{p+1}^2\right).$$

This article presents sharp estimates for Toeplitz determinants $T_2(p+1)$ and $T_3(p)$ for functions that belong to the $\mathcal{S}_{p,q}^*(\psi)$ and $\mathcal{C}_{p,q}(\psi)$ classes. The functions $K_{p,q}(\psi)$ and $H_{p,q}(\psi)$, which are defined as follows:

$$\frac{1}{[p]_q}\frac{z\mathcal{D}_q\left(K_{p,q}\right)\psi(z)}{\left(K_{p,q}\right)\psi(z)}=\psi(\iota z),\left(z\in\mathcal{U}; K_{p,q}\psi(z)\in\mathcal{A}_p\right), \tag{1.4}$$

and

$$\frac{1}{[p]_q}\left(1+\frac{z\mathcal{D}_q\left(\mathcal{D}_q H_{p,q}\psi(z)\right)}{\mathcal{D}_q H_{p,q}\psi(z)}\right)=\psi(\iota z),\left(z\in\mathcal{U}; H_{p,q}\psi(z)\in\mathcal{A}_p\right), \tag{1.5}$$

respectively, these functions belong to the classes $\mathcal{S}_{p,q}^*(\psi)$ and $\mathcal{C}_{p,q}(\psi)$. We shall employ these functions to demonstrate the sharpness of our results in certain instances. It is a classical result due to Grenander and Szegö[16] that if a function $s\in\mathcal{P}$ is given by $s(z)=1+c_1z+c_2z^2+c_3z^3+\cdots$, we have $|c_n|\le 2$.The main results of the present work are derived using this well-known estimate by relating the coefficients of functions in the classes $\mathcal{S}_{p,q}^*(\psi)$ and $\mathcal{C}_{p,q}(\psi)$ with those of functions belonging to the class $\mathcal{P}$. Furthermore, we shall make use of the Fekete-Szegö functional estimates for the classes $\mathcal{S}_{p,q}^*(\psi)$ and $\mathcal{C}_{p,q}(\psi)$. Ma and Minda [29] used the symmetry of the image of φ to guarantee that the coefficients of φ are real, and we follow this assumption for the first two coefficients. In addition, they employed the univalence property in defining the function s_1, where

$$s_1(z)=\frac{1-\varphi^{-1}(s(z))}{1+\varphi^{-1}(s(z))}.$$

The above condition can be relaxed by adopting the definition of s_1 given in equation (2.2).

Toeplitz Coefficient bounds

In this section the sharp bound for $T_2(p+1)=a_{p+1}^2-a_{p+2}^2$ is given by Theorem 3 and Theorem 8 for the functions $\in\mathcal{S}_{p,q}^*(\psi)$ and $f\in\mathcal{C}_{p,q}(\psi)$ respectively.

- **Theorem 3.** If $f\in\mathcal{S}_{p,q}^*(\psi)$ and $\varphi(z)=1+B_1z+B_2z^2+\cdots$ with $0<B_1\le\left|B_2+\frac{B_1^2[p]_q}{q^p}\right|$, then the Toeplitz determinant $T_2(p+1)$ satisfies the sharp bound:

$$|T_2(p+1)|\le\frac{[p]_q^2}{q^{2p}[2]_q^2}\left(\frac{[p]_q}{q^p}B_1^2+B_2\right)^2+\frac{[p]_q^2}{q^{2p}}B_1^2.$$

Proof. Since $f\in\mathcal{S}_{p,q}^*(\psi)$, there is a function w in the class Ω of Schwarz functions satisfying that

$$\frac{z\mathcal{D}_q f(z)}{[p]_q f(z)}=\varphi(w(z)). \tag{2.1}$$

Associated with the function w, define the function $s_1 : \mathcal{U} \to \mathbb{C}$ by

$$s_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots, \tag{2.2}$$

so that

$$w(z) = \frac{s_1(z) - 1}{s_1(z) + 1} = \frac{1}{2} c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2} c_1^2\right) z^2 + \cdots. \tag{2.3}$$

Clearly, the function s_1 is analytic in $\mathcal{U}$ with $s_1(0) = 1$. Since $w \in \Omega$, it follows that $s_1 \in \mathcal{P}$. Using (2.3) and the Taylor series of ψ given by $\psi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, we get

$$\psi(w(z)) = 1 + \frac{1}{2} B_1 c_1 z + \left(\frac{1}{2} B_1 \left(c_2 - \frac{1}{2} c_1^2\right) + \frac{1}{4} B_2 c_1^2\right) z^2 + \cdots. \tag{2.4}$$

Since $f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$, the taylor series expansion of the function $\frac{z\mathcal{D}_q f(z)}{[p]_q f(z)}$ is given by

$$\frac{z\mathcal{D}_q f(z)}{[p]_q f(z)} = 1 + \frac{q^p a_{p+1}}{[p]_q} z + \left(\frac{q^p [2]_q}{[p]_q} a_{2+p} - \frac{q^p}{[p]_q} a_{1+p}^2\right) z^2 + \cdots. \tag{2.5}$$

Using (2.1), (2.4), (2.5) the coefficients a_{p+1} and a_{p+2} can be formulated using the coefficients c_i corresponding to $s \in \mathcal{P}$ and B_i of φ as shown below:

$$a_{p+1} = \frac{[p]_q}{2q^p} B_1 c_1 \tag{2.6}$$

and

$$a_{p+2} = \frac{[p]_q}{4[2]_q q^p}\left(\left(\frac{[p]_q}{q^p} B_1^2 - B_1 + B_2\right) c_1^2 + 2B_1 c_2\right). \tag{2.7}$$

The equations (2.6) and (2.7) (see Ali et al.[4] for a general result for p-valent functions) readily shows that

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \begin{cases} \frac{[p]_q}{[2]_q q^p}\left(B_2 + \frac{B_1^2 [p]_q}{q^p} - \frac{\mu [p]_q [2]_q B_1^2}{q^p}\right); \frac{\mu [p]_q [2]_q B_1^2}{q^p} \le B_2 - B_1 + \frac{B_1^2 [p]_q}{q^p}, \\ \frac{B_1 [p]_q}{[2]_q q^p};\ B_2 - B_1 + \frac{B_1^2 [p]_q}{q^p} \le \frac{\mu [p]_q [2]_q B_1^2}{q^p} \le B_2 + B_1 + \frac{B_1^2 [p]_q}{q^p}, \\ \frac{[p]_q}{[2]_q q^p}\left(-B_2 - \frac{B_1^2 [p]_q}{q^p} + \frac{\mu [p]_q [2]_q B_1^2}{q^p}\right); \frac{\mu [p]_q [2]_q B_1^2}{q^p} \ge B_2 + B_1 + \frac{B_1^2 [p]_q}{q^p}. \end{cases} \tag{2.8}$$

Since $|c_n| \le 2$. the equation (2.6) shows that

$$\left|a_{p+1}\right| \le \frac{[p]_q}{q^p} B_1 \tag{2.9}$$

and $0 < B_1 \le \left|B_2 + \frac{B_1^2 [p]_q}{q^p}\right|$, the equation (2.7) readily yields

$$\left|a_{p+2}\right| \le \frac{[p]_q}{q^p [2]_q}\left|\frac{[p]_q}{q^p} B_1^2 + B_2\right|. \tag{2.10}$$

Using these estimates for the second and third coefficients given in (2.9) and (2.10) , we have

$$|T_2(p+1)| = \left|a_{p+2}^2 - a_{p+1}^2\right| \le \left|a_{p+2}\right|^2 + \left|a_{p+1}\right|^2 \le \frac{[p]_q^2}{q^{2p}[2]_q^2}\left(\frac{[p]_q}{q^p} B_1^2 + B_2\right)^2 + \frac{[p]_q^2}{q^{2p}} B_1^2.$$

The result is sharp for the function $K_{p,q}\psi(z)$ given by (1.4). This function $K_{p,q}\psi(z)$ has the Taylor series given by

$$K_{p,q}\psi(z) = z^p - \frac{[p]_q B_1}{q^p} z^{p+1} + \frac{[p]_q}{q^p [2]_q}\left(\frac{B_1^2 [p]_q}{q^p} + B_2\right) z^{p+2} \cdots.$$

The Taylor series can be obtained by noting that $K_{p,q}\psi(z)$ corresponds to the function f given by (2.1) when $w(z) = iz$. In this case, $p_1(z) = 1 + 2iz - 2z^2 + \cdots$. With $c_1 = 2i$ and $c_2 = -2$, we get $a_{p+1} = \frac{-\iota[p]_q}{q^p} B_1$ and $a_{p+2} = -\frac{[p]_q}{q^p[2]_q}\left(\frac{[p]_q}{q^p} B_1^2 + B_2\right)$. Clearly, for the function $K_{p,q}\psi(z)$, we have

$|T_2(p+1)| = |a_{p+1}^2 - a_{p+2}^2| \le |a_{p+1}|^2 + |a_{p+2}|^2 = \frac{[p]_q^2}{q^{2p}[2]_q^2}\left(\frac{[p]_q}{q^p} B_1^2 + B_2\right)^2 + \frac{[p]_q^2}{q^{2p}} B_1^2$

proving the sharpness. □

- **Remark 4.** In the classical limit $q \to 1^-$, we have $[p]_q \to p$ and $q^p \to 1$. Then Theorem 3 reduces to:

$$|T_2(p+1)| \le \frac{p^2}{4}(pB_1^2 + B_2)^2 + p^2B_1^2.$$

For the special case $p = 1$, this further simplifies to the well-known bound for univalent starlike functions $S^*(\psi)$:

$$|a_3^2 - a_2^2| \le \frac{1}{4}(B_1^2 + B_2)^2 + B_1^2.$$

Thus, Theorem 3 generalizes and unifies several classical results in geometric function theory.

- **Corollary 5.** Under the same hypotheses as Theorem 3, the difference between the absolute values of the first two non-zero coefficients satisfies:

$$\left||a_{p+2}| - |a_{p+1}|\right| \le \frac{[p]_q}{q^p[2]_q}\left|\frac{[p]_q}{q^p} B_1^2 + B_2\right| + \frac{[p]_q}{q^p} B_1.$$

Proof. By the triangle inequality:

$$\left||a_{p+2}| - |a_{p+1}|\right| \le |a_{p+2}| + |a_{p+1}|.$$

Using the bounds $|a_{p+1}| \le \frac{[p]_q}{q^p} B_1$ and $|a_{p+2}| \le \frac{[p]_q}{q^p[2]_q}\left|\frac{[p]_q}{q^p} B_1^2 + B_2\right|$ from Theorem 3, we obtain the desired inequality. □

- **Corollary 6.** Under the same hypotheses as Theorem 3, the coefficient a_{p+2} satisfies the sharp bound:

$$|a_{p+2}| \le \frac{[p]_q}{q^p[2]_q}\left|\frac{[p]_q}{q^p} B_1^2 + B_2\right|.$$

Proof. Sharpness is attained by the extremal function $K_{p,q}\psi(z)$ given in Theorem 3. □

- **Corollary 7.** Under the same hypotheses as Theorem 3, the ratio of the second non-zero coefficient to the first non-zero coefficient satisfies:

$$\left|\frac{a_{p+2}}{a_{p+1}}\right| \le \frac{1}{[2]_q} \cdot \frac{\left|\frac{[p]_q}{q^p} B_1^2 + B_2\right|}{B_1},$$

provided $a_{p+1} \ne 0$ and $B_1 > 0$.

Proof. From Theorem 3, we have the sharp bounds:

$$|a_{p+2}| \le \frac{[p]_q}{q^p[2]_q}\left|\frac{[p]_q}{q^p} B_1^2 + B_2\right|,$$

and from equation 2.8 in the proof of Theorem 3, the coefficient $|a_{p+1}|$ can be arbitrarily close to $\frac{[p]_q}{q^p} B_1$ (by taking $|c_1| = 2$). Therefore:

$$\left|\frac{a_{p+2}}{a_{p+1}}\right| = \frac{|a_{p+2}|}{|a_{p+1}|} \le \frac{\frac{[p]_q}{q^p[2]_q}\left|\frac{[p]_q}{q^p}B_1^2 + B_2\right|}{\frac{[p]_q}{q^p}B_1}.$$

Simplifying the common factor $\frac{[p]_q}{q^p}$ gives:

$$\left|\frac{a_{p+2}}{a_{p+1}}\right| \le \frac{1}{[2]_q} \cdot \frac{\left|\frac{[p]_q}{q^p}B_1^2 + B_2\right|}{B_1}.$$

This completes the proof. □

- **Remark 1.** The following 3D graph shows that how the bound of the Toeplitz determinant $T_2(p+1)$ changes with respect to the parameters q and B_1. The surface clearly shows that when the value of q decreases, the bound becomes larger, which indicates that the q-deformation increases the size of the Toeplitz determinant. Similarly, as B_1 increases, the bound also grows, showing that the first coefficient of the subordinant function ψ plays an important role in controlling the determinant. Thus, the graph visually supports the analytic formula obtained in Theorem 3.
- **Remark 2.** The second 3D surface plot illustrates the variation of the Toeplitz determinant bound with respect to parameters q and B_2. For a fixed value of B_1, the graph shows that the bound increases as B_2 becomes larger, which is consistent with the analytic expression of the coefficient a_{p+2} in Theorem 3. In addition, the surface rises sharply for smaller values of q, indicating that the q-deformation amplifies the determinant bound. Thus, the graph provides a visual confirmation of the theoretical behavior derived earlier.
- **Theorem 8.** If $f \in \mathcal{C}_{p,q}(\varphi)$ and $\varphi(z) = 1 + B_1z + B_2z^2 + \cdots$ with $0 < B_1 \le \left|B_2 + \frac{[p]_q}{q^p}B_1^2\right|$, then the Toeplitz determinant $T_2(p+1)$ satisfies the sharp bound given by

$$|T_2(p+1)| \le \frac{[p]_q^4}{[2+p]_q^2 q^{2p}[2]_q^2}\left(\frac{[p]_qB_1^2}{q^p} + B_2\right)^2 + \frac{[p]_q^4}{q^{2p}[p+1]_q^2}B_1^2.$$

Proof. Let $f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}z^{n+p}$ and $\varphi(z) = 1 + B_1z + B_2z^2 + \cdots$. Since $f \in \mathcal{C}_{p,q}(\varphi)$, a function w in the class Ω of Schwarz functions exists such that definition 2 can be written as

$$\frac{1}{[p]_q}\left(1 + \frac{qz\mathcal{D}_q(\mathcal{D}_qf(z))}{\mathcal{D}_qf(z)}\right) = \varphi(w(z)). \tag{2.11}$$

Using the taylor series expansion of the function $f(z)$, we obtain

$$\frac{1}{[p]_q}\left(1 + \frac{qz\mathcal{D}_q(\mathcal{D}_qf(z))}{\mathcal{D}_qf(z)}\right) = 1 + \left(\frac{q^p[p+1]_qa_{p+1}}{[p]_q^2}\right)z$$

(2.12)

$$+\left(\frac{[2]_q[p+2]_qq^pa_{p+2}}{[p]_q^2} - \frac{[p+1]^2q^pa_{p+1}^2}{[p]_q^3}\right)z^2 + \cdots.$$

Then using (2.11), (2.12) and (2.4), the coefficients a_{p+1} and a_{p+2} can be expressed as a function of the coefficients c_i of $p \in \mathcal{P}$ given by

$$a_{p+1} = \frac{[p]_q^2}{2q^p[p+1]_q}B_1c_1$$

and

$$a_{p+2} = \frac{[p]_q^2}{q^p[2]_q[p+2]_q}\left[\frac{B_1c_2}{2} + \frac{1}{4}\left(\frac{[p]_q}{q^p}B_1^2 - B_1 + B_2\right)c_1^2\right].$$

In view of the well-known estimate $|c_n| \leq 2$ for functions p_1 with positive real part, we conclude that

$$\left|a_{p+1}\right| \leq \frac{[p]_q^2 B_1}{q^p [p+1]_q}. \tag{2.13}$$

Considering the function $f \in \mathcal{C}_{p,q}(\varphi)$, one can easily deduce that

$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq$

$$\begin{cases} \dfrac{[p]_q^2}{\left[p+2]_q q^p\right]_q}\left(B_2 + \dfrac{[p]_q}{q^p} B_1^2 - \dfrac{\mu [p]_q^2 [p+2]_q [2]_q}{q^p [p+1]_q^2} B_1^2\right); \ \mu [p]_q^2 [p+2]_q B_1^2 \leq \dfrac{q^p [1+p]_q^2}{[2] q_q}\left(B_2 - B_1 + \dfrac{[p]_q}{q^p} B_1^2\right), \\ \dfrac{[p+2]_q q^p}{[p+2]_q} B_1; \ \dfrac{q^p [1+p]_q^2}{[2]_q}\left(B_2 - B_1 + \dfrac{[p]_q}{q^p} B_1^2\right) \leq \mu [p]_q^2 [p+2]_q B_1^2 \leq \dfrac{q^p [1+p]_q^2}{[2]_q}\left(B_2 + B_1 + \dfrac{[p] q}{q^p} B_1^2\right), \\ \dfrac{[p+2]_q \, {}^q p [2]_q}{[2]}\left(-B_2 - \dfrac{[p]_q}{q^p} B_1^2 + \dfrac{\mu [p]_q^2 [p+2]_q [2] q}{q^p [p+1]_q^2} B_1^2\right); \ \dfrac{q^p [1+p]_q^2}{[2]_q}\left(B_2 + B_1 + \dfrac{[p]_q}{q^p} B_1^2\right) \leq \mu [p]_q^2 [p+2]_q B_1^2. \end{cases} \tag{2.14}$$

Since $B_1 \leq \left|B_2 + \frac{B_1^2 [p]_q}{q^p}\right|$, the inequality (2.8) readily gives

$$\left|a_{p+2}\right| \leq \frac{[p]_q^2 \left(\dfrac{[p]_q B_1^2}{q^p} + B_2\right)}{[2]_q q^p [p+2]_q}. \tag{2.15}$$

Using the bound for a_{p+1} and a_{p+2} given respectively, by (2.13) and (2.15), we get

$$\left|a_{p+1}^2 - a_{p+2}^2\right| \leq \frac{[p]_q^4}{[p+2]_q^2 q^{2p} [2]_q^2}\left(\frac{[p]_q B_1^2}{q^p} + B_2\right)^2 + \frac{[p]_q^4}{q^{2p} [p+1]_q^2} B_1^2. \tag{2.16}$$

The result is sharp for the function $H_{p,q}(\psi)$ defined in (1.5). Indeed, for this function $H_{p,q}(\psi)$, we have $a_{p+1} = \frac{\iota [p]_q^2 B_1}{q^p [p+1]_q}$ and $a_{p+2} = -\frac{[p]_q^2}{[2]_q q^p [p+2]_q}\left(\frac{[p]_q B_1^2}{q^p} + B_2\right)$ and hence

$$\left|a_{p+1}^2 - a_{p+2}^2\right| = \frac{[p]_q^4}{[p+2]_q^2 q^{2p} [2]_q^2}\left(\frac{[p]_q B_1^2}{q^p} + B_2\right)^2 + \frac{[p]_q^4}{q^{2p} [p+1]_q^2} B_1^2$$

proving the sharpness of the result. □

- **Remark 9.** In the classical limit $q \to 1^-$, we have $[p]_q \to p, [p+1]_q \to p+1$, and $[2+p]_q \to p+2$. Then Theorem 8 reduces to:

$$|T_2(p+1)| \leq \frac{p^4}{(p+2)^2 \cdot 4}(pB_1^2 + B_2)^2 + \frac{p^4}{(p+1)^2} B_1^2.$$

For $p = 1$, this further simplifies to the well-known bound for convex functions $C(\varphi)$:

$$|a_3^2 - a_2^2| \leq \frac{1}{36}(B_1^2 + B_2)^2 + \frac{1}{4} B_1^2.$$

Thus, Theorem 8 generalizes the classical convex function result.

- **Corollary 10.** Under the same hypotheses as Theorem 8 (convex) and Theorem 3 (starlike), the ratio of the Toeplitz determinant bounds for convex and starlike functions is:

$$\frac{\left|T_2^{\text{convex}}(p+1)\right|}{\left|T_2^{\text{starlike}}(p+1)\right|} \leq \frac{[p]_q^4}{[p+1]_q^2} \cdot \frac{[2]_q^2}{[2+p]_q^2}.$$

In the classical limit $q \to 1^-$, this ratio becomes:

$$\frac{\left|T_2^{\text{convex}}(p+1)\right|}{\left|T_2^{\text{starlike}}(p+1)\right|} \leq \frac{4p^4}{(p+1)^2 (p+2)^2}.$$

For $p = 1$, this gives $\frac{4}{36} = \frac{1}{9}$, showing that convex functions have Toeplitz determinants at most one-ninth of starlike functions.

Proof. Divide the bound from Theorem 8 by the bound from Theorem 3 and simplify. Theorems 11 and 15 give the sharp bound for the Toeplitz determinant $T_3(p)$ for functions, respectively, in the classes $\mathcal{S}^*_{p,q}(\psi)$ and $\mathcal{C}_{p,q}(\psi)$. □

- **Theorem 11.** If $f \in \mathcal{S}^*_{p,q}(\varphi)$ and $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, with $B_1 > 0$ and $B_1 - \frac{B_1^2[p]_q}{q^p} \leq B_2 \leq \frac{B_1^2[p]_q}{q^p}\left(2[2]_q - 1\right) - B_1$, then the Toeplitz determinant $T_3(p)$ satisfies the sharp bound:

$$|T_3(p)| \leq 1 + \frac{2[p]_q^2 B_1^2}{q^{2p}} + \frac{[p]_q^2}{q^{2p}[2]_q^2}\left(\frac{[p]_q B_1^2}{q^p} + B_2\right)\left(-B_2 + \frac{[p]_q}{q^p}\left(2[2]_q - 1\right)B_1^2\right).$$

Proof. Since

$$T_3(p) = \begin{vmatrix} 1 & a_{p+1} & a_{p+2} \\ a_{p+1} & 1 & a_{p+1} \\ a_{p+2} & a_{p+1} & 1 \end{vmatrix} = 1 - 2a_{p+1}^2 - a_{p+2}\left(a_{p+2} - 2a_{p+1}^2\right),$$

it follows that

$$|T_3(p)| \leq 1 + 2\left|a_{p+1}\right|^2 + \left|a_{p+2}\right|\left|a_{p+2} - 2a_{p+1}^2\right|. \tag{2.17}$$

Since $0 \leq B_1 \leq B_2 + \frac{B_1^2[p]_q}{q^p}$, the inequality (2.10) gives

$$\left|a_{p+2}\right| \leq \frac{[p]_q}{[2]_q q^p}\left|\frac{[p]_q}{q^p}B_1^2 + B_2\right|. \tag{2.18}$$

Since $B_1 + B_2 \leq \frac{B_1^2[p]_q}{q^p}\left(2[2]_q - 1\right)$, the equation (2.8) readily yields

$$\left|a_{p+2} - 2a_{p+1}^2\right| \leq \frac{[p]_q}{[2]_q q^p}\left(\frac{[p]_q}{q^p}\left(2[2]_q - 1\right)B_1^2 - B_2\right). \tag{2.19}$$

Using these estimates for the second and third coefficients given in (2.9) and (2.18), and the bound for $a_{p+2} - 2a_{p+1}^2$ given by (2.19) in (2.17), we obtain

$$|T_3(p)| \leq 1 + \frac{2[p]_q^2 B_1^2}{q^{2p}} + \frac{[p]_q^2}{q^{2p}[2]_q^2}\left(\frac{[p]_q B_1^2}{q^p} + B_2\right)\left(-B_2 + \frac{[p]_q}{q^p}\left(2[2]_q - 1\right)B_1^2\right).$$

The result is sharp for the function $K_p(\psi)$ given by (2.1). For this function $K_p(\psi)$, we have $a_{p+1} = \frac{-\iota[p]_q}{q^p}B_1$ and $a_{p+2} = -\frac{[p]_q}{q^p[2]_q}\left(\frac{[p]_q}{q^p}B_1^2 + B_2\right)$ and

$$\begin{aligned} |T_3(p)| &= 1 - 2a_{p+1}^2 - a_{p+2}\left(a_{p+2} - 2a_{p+1}^2\right) \\ &= 1 + \frac{2[p]_q^2 B_1^2}{q^{2p}} + \frac{[p]_q^2}{q^{2p}[2]_q^2}\left(\frac{[p]_q B_1^2}{q^p} + B_2\right)\left(-B_2 + \frac{[p]_q}{q^p}\left(2[2]_q - 1\right)B_1^2\right), \end{aligned}$$

proving the sharpness of our result. □

- **Corollary 12.** For $p = 1$, Theorem 11 reduces to the class $\mathcal{S}^*_{1,q}(\varphi)$ of q-starlike functions. The Toeplitz determinant $T_3(1)$ satisfies:

$$|T_3(1)| \leq 1 + \frac{2B_1^2}{q^2} + \frac{1}{q^2[2]_q^2}\left(\frac{B_1^2}{q} + B_2\right)\left(-B_2 + \frac{1}{q}\left(2[2]_q - 1\right)B_1^2\right).$$

In the classical limit $q \to 1^-$, this becomes $|T_3(1)| \leq 1 + 2B_1^2 + \frac{1}{4}(B_1^2 + B_2)(-B_2 + 3B_1^2)$, recovering a known result for univalent starlike functions.

- **Remark 13.** Under the hypotheses of Theorem 11, the following inequality connects $T_3(p)$ with $T_2(p+1)$ from Theorem 3:

$|T_3(p)| \leq 1 + |T_2(p+1)| + \frac{[p]_q^2}{q^{2p}[2]_q^2}\left(\frac{[p]_q}{q^p}B_1^2 + B_2\right)\left(-B_2 + \frac{[p]_q}{q^p}(2[2]_q - 1)B_1^2 - \frac{[p]_q}{q^p}B_1^2\right).$

Thus, Theorem 11 generalizes Theorem 3 to higher-order Toeplitz determinants. **Theorem 14.** In the classical limit $q \to 1^-$, Theorem 3 reduces to:

$$|T_3(p)| \leq 1 + 2p^2B_1^2 + \frac{p^2}{4}(pB_1^2 + B_2)(-B_2 + 3pB_1^2).$$

For large p, the leading asymptotic behavior is:

$$|T_3(p)| \sim \frac{3p^4B_1^4}{4} \text{ as } p \to \infty.$$

Proof. As $q \to 1^-$, we have $[p]_q \to p, [2]_q \to 2$, and $q^p \to 1$. Substituting gives the first result. For large p, the dominant term is $\frac{p^2}{4}(pB_1^2)(3pB_1^2) = \frac{3p^4B_1^4}{4}$. □

- **Remark 3.** The three-dimensional surface plot associated with Theorem 11 provides a visual representation of the behaviour of the Toeplitz determinant $T_3(p)$ with respect to the parameters q and B_2. For a fixed value of B_1, the bound exhibits a monotonic increase as B_2 grows, reflecting the direct contribution of the second coefficient of the subordinating function φ to the magnitude of the determinant. Furthermore, the surface indicates that smaller values of the parameter q lead to a significant escalation in the bound, thereby demonstrating the influence of the q-deformation on the coefficient structure of functions in the class $\mathcal{S}_{p,q}^*(\varphi)$. The graphical behavior aligns well with the analytic estimate obtained in Theorem 11, thereby offering a complementary geometric insight into the dependence of the Toeplitz determinant on the underlying parameters.
- **Theorem 15.** If $f \in \mathcal{C}_{p,q}(\varphi)$ and $\varphi(z) = 1 + B_1z + B_2z^2 + \cdots$, with $B_1 > 0$ and $B_1 - pB_1^2 \leq B_2 \leq 2pB_1^2 - B_1$, then the Toeplitz determinant $T_3(p)$ satisfies the sharp bound:

$$\begin{aligned}|T_3(p)| \leq 1 &+ \frac{2[p]_q^4}{q^{2p}[p+1]_q^2}B_1^2\\ &+ \frac{[p]_q^4}{[2+p]_q^2q^{2p}[2]_q^2}\left(\frac{[p]_q}{q^p}B_1^2 + B_2\right)\left[\frac{[p]_q}{q^p}\left(\frac{2[p]_q^2[2+p]_q[2]_q}{[1+p]_q^2} - 1\right)B_1^2 - B_2\right].\end{aligned}$$

Proof. The given conditions on B_1 and B_2 is the same as $0 \leq B_1 \leq B_2 + \frac{B_1^2[p]_q}{q^p}$ and, $B_1 + B_2 \leq \frac{[p]_qB_1^2}{q^p}\left(\frac{\mu[2]_q[p]_q[p+2]_q}{[p+1]_q^2} - 1\right)$. Since $B_1 \leq B_2 + \frac{B_1^2[p]_q}{q^p}$, the inequality (2.14) gives

$$|a_{p+1}| \leq \frac{[p]_q^2B_1}{q^p[p+1]_q}, \tag{2.20}$$

and

$$|a_{p+2}| \leq \frac{[p]_q^2\left(\frac{[p]_q}{q^p}B_1^2 + B_2\right)}{q^p[2]_q[p+2]_q}. \tag{2.21}$$

Since $B_1 + B_2 \leq \frac{[p]_qB_1^2}{q^p}\left(\frac{\mu[2]_q[p]_q[p+2]_q}{[p+1]_q^2} - 1\right)$, the inequality (2.14) gives

$$|a_{p+2} - 2a_{p+1}^2| \leq \frac{[p]_q^2\left(\frac{[p]_q}{q^p}\left(\frac{2[p]_q^2[2+p]_q[2]_q}{[p+1]_q^2} - 1\right)B_1^2 - B_2\right)}{q^p[2]_q[p+2]_q}. \tag{2.22}$$

Using the bound for a_{p+1} and a_{p+2} given by (2.13) and (2.20) and the bound for $a_{p+2} - 2a_{p+1}^2$ given by (2.22) in (2.17), we get the desired result.

The result is sharp for the function H_φ defined in (1.5). Indeed, for this function H_φ, we have $a_{p+1} = \frac{[p]_q^2B_1\iota}{q^p[p+1]_q}$ and $a_{p+2} = \frac{-[p]_q^2\left[\frac{[p]_q}{q^p}B_1^2+B_2\right)}{q^p[2]_q[p+2]_q}$ and hence

$$|T_3(p)| = 1 - 2a_{p+1}^2 - a_{p+2}\left(a_{p+2} - 2a_{p+1}^2\right) = 1 + \frac{2[p]_q^4}{q^{2p}[p+1]_q^2}B_1^2 + \frac{[p]_q^4}{[2+p]_q^2 q^{2p}[2]_q^2}\left(\frac{[p]_q}{q^p}B_1^2 + B_2\right)\left[\frac{[p]_q}{q^p}\left(\frac{2[p]_q^2[P+2]_q[2]_q}{[p+1]_q^2} - 1\right)B_1^2 - B_2\right],$$

proving the sharpness of the result.

Taking the different values of $\psi(z)$, we obtain the Toeplitz bounds $T_2(p+1)$ and $T_3(p)$ for starlike and convex classes $\mathcal{S}_{p,q}^*(\psi)$ and $\mathcal{C}_{p,q}(\psi)$.

- **Corollary 16.** In the classical limit $\mathbf{q} \to \mathbf{1}^-$, Theorem 15 reduces to:

$$|\mathbf{T_3(p)}| \le \mathbf{1} + \frac{\mathbf{2p^4}}{(\mathbf{p+1})^2}\mathbf{B_1^2} + \frac{\mathbf{p^4}}{(\mathbf{p+2})^2 \cdot \mathbf{4}}\left(\mathbf{pB_1^2} + \mathbf{B_2}\right)\left(\mathbf{p}\left(\frac{\mathbf{2p^2(p+2)\cdot 4}}{(\mathbf{p+1})^2} - \mathbf{1}\right)\mathbf{B_1^2} - \mathbf{B_2}\right).$$

For $\mathbf{p} = \mathbf{1}$, this gives $|\mathbf{T_3(1)}| \le \mathbf{1} + \frac{1}{2}\mathbf{B_1^2} + \frac{1}{36}\left(\mathbf{B_1^2} + \mathbf{B_2}\right)\left(\mathbf{5B_1^2} - \mathbf{B_2}\right)$, a new result for univalent convex functions.

- **Theorem 17.** For fixed $\mathbf{q}$ and $\mathbf{B_1}, \mathbf{B_2}$, as $\mathbf{p} \to \infty$, the bound in Theorem 15 behaves asymptotically as:

$$|\mathbf{T_3(p)}| \sim \frac{\mathbf{C}}{\mathbf{q^{2p}}} \cdot \mathbf{p^8} \text{ as } \mathbf{p} \to \infty,$$

where $\mathbf{C}$ is a constant depending on $\mathbf{q}, \mathbf{B_1}, \mathbf{B_2}$. This shows exponential growth in $\mathbf{p}$ due to the $\mathbf{q^{2p}}$ factor. □

Exploring The Special Cases

$$\boldsymbol{\psi}(\mathbf{z}) = \frac{\mathbf{1+Az}}{\mathbf{1+Bz}}, (-\mathbf{1} \le \mathbf{B} < \mathbf{A} \le \mathbf{1}, \mathbf{z} \in \boldsymbol{\mathcal{U}})$$

The Ma-Minda classes of starlike and convex functions contain several important subclasses that have been widely studied by various authors (see, for example, Kargar et al.[24] and Mahzoon[30]). Theorems 3 to 15 provide sharp bounds for $|T_2(p+1)|$ and $|T_3(p)|$ within certain subclasses of these families. These subclasses, referred to as Janowski starlike functions and Janowski convex functions, were originally introduced and analyzed by Janowski[21]. The series expansion of $\psi(z) = \frac{1+Az}{1+Bz}$ leads to the following representation:

$$\psi(z) := \frac{1+Az}{1+Bz} = 1 + (A-B)z + B(B-A)z^2 + B^2(A-B)z^3 + \cdots,$$

which implies $B_1 = (A-B)$ and $B_2 = -B(A-B)$. If $|A-2B| \ge 1$, then for $f \in \mathcal{S}_{p,q}^*[A,B]$,

$$|T_2(p+1)| \le \frac{[p]_q^2(A-B)^2}{q^{2p}[2]_q^2}\left[\left(\frac{[p]_q}{q^p}(A-B) - B\right)^2 + [2]_q^2\right]$$

and for $f \in \mathcal{C}_{p,q}[A,B]$, we have

$$|T_2(p+1)| \le \frac{[p]_q^4(A-B)^2}{q^{2p}[2]_q^2[2+p]_q^2}\left[\frac{[2]_2^2[2+p]_q^2}{[1+p]_q^2} + \frac{[p]_q^2}{q^{2p}}A^2 - \frac{2[p]_q}{q^p}\left(\frac{[p]_q}{q^p} + 1\right)AB + B^2\left(1 + \frac{[p]_q^2}{q^{2p}} + \frac{2[2]_q}{q^p}\right)\right].$$

If $B \le \min\{(A-1)/2, (3A-1)/2\}$, then for $f \in \mathcal{S}_{p,q}^*[A,B]$, we have

$$|T_3(p)| \le 1 + \frac{[p]_q^2(A-B)^2}{q^{2p}[2]_q^2}\left[2[2]_q^2 + \left(\frac{[p]_q}{q^p}(A-B) - B\right)\left(\frac{[p]_q}{q^p}\left(2[2]_q - 1\right)(A-B) + B\right)\right].$$

If $(A+B) \ge 0$ and $B \le (A-1)/2$, then for $f \in \mathcal{C}_{p,q}[A,B]$, then

$$|T_3(p)| \le 1+\frac{2[p]_q^4(A-B)^2}{q^{2p}[p+1]_q^2}$$
$$+\frac{[p]_q^4(A-B)^2}{[p+2]_q^2[2]_q^2}\left[\left(\frac{[p]_q}{q^p}(A-B)-B\right)\left(\frac{[p]_q}{q^p}\left(\frac{2[2]_q[P+2]_q[p]_q^2}{[p+1]_q^2}-1\right)(A-B)+B\right)\right].$$

$\boldsymbol{\psi(z)=\alpha+(1-\alpha)e^z, (0\le\alpha<1)}$

The Taylor series expansion of ψ is given by

$$\psi(z)=\alpha+(1-\alpha)e^z=1+(1-\alpha)z+\frac{1}{2}(1-\alpha)z^2+\frac{1}{6}(1-\alpha)z^3+\cdots$$

shows that $B_1=(1-\alpha)$ and $B_2=(1-\alpha)/2$. For $0\le\alpha\le 1/2$, then for $f\in\mathcal{S}_{p,q}^*[\alpha+(1-\alpha)e^z]$,

$$|T_2(p+1)| \le \frac{[p]_q^2(1-\alpha)^2)}{q^{2p}[2]_q^2}\left[\left(\frac{[p]_q}{q^p}(1-\alpha)+\frac{1}{2}\right)^2+[2]_q^2\right]$$

and

$$|T_3(p)| \le 1+\frac{[p]_q^2(1-\alpha)^2}{q^{2p}[2]_q^2}\left[2[2]_q^2+\left[\frac{[p]_q}{q^p}(1-\alpha)+\frac{1}{2}\right]\left[\frac{[p]_q}{q^p}\big(2[2]_q-1\big)(1-\alpha)-\frac{1}{2}\right]\right].$$

For $f\in\mathcal{C}_{p,q}[\alpha+(1-\alpha)e^z]$, we get

$$|T_2(p+1)| \le \frac{[p]_q^4(1-\alpha)^2}{q^{2p}[2]_q^2}\left[\frac{1}{[p+2]_q^2}\left(\frac{[p]_q}{q^p}(1-\alpha)+\frac{1}{2}\right)^2+\frac{[2]_q^2}{[p+1]_q^2}\right]$$

and

$$|T_3(p)| \le 1+\frac{[p]_q^4(1-\alpha)^2}{q^{2p}[2]_q^2}\left[\frac{2[2]_q^2}{[p+1]_q^2}\right.$$
$$\left.+\frac{1}{[p+2]_q^2}\left[\frac{[p]_q}{q^p}(1-\alpha)+\frac{1}{2}\right]\left[\frac{[p]_q}{q^p}\left(\frac{2[2]_q[p]_q^2[p+2]_q}{[p+1]_q^2}-1\right)(1-\alpha)-\frac{1}{2}\right]\right].$$

For $\alpha=0, p=1$ and $q\to 1^-, f\in\mathcal{S}^*(e^z)$, we obtain the bounds $|T_2(2)|\le 25/16\approx 1.5625$ and $|T_3(1)|\le 63/16\approx 3.9375$. Similarly, $\alpha=0, p=1$ and $q\to 1^-$, $f\in\mathcal{C}(e^z)$, we have $|T_2(2)|\le 5/16\approx 0.3125$ and $|T_3(1)|\le 25/16\approx 1.5625$.

3.3 $\boldsymbol{\psi(z)=\psi_c(z)=1+(4/3)z+(2/3)z^2, (z\in\mathcal{U})}$

A function f belongs to the class $\mathcal{S}_{p,q}^*(\psi_c(z))$ if $\frac{z\mathcal{D}_qf(z)}{[p]_qf(z)}$ lies in the region Ω_c bounded by the cardioid i.e.,

$$\psi_c(\mathcal{U}):=\{x+iy:(9x^2+9y^2-18x+5)^2-16(9x^2+9y^2-6x+1)=0\}.\text{The}$$

class $\mathcal{C}_{p,q}:=\mathcal{C}_{p,q}(\psi_c(z))$ serves as the convex analogous of the class mentioned above. Geometrically, this means that a function f belongs to the class $\mathcal{C}_{p,q}(\psi_c(z))$ if $\frac{1}{[p]_q}\left(1+\frac{qz\mathcal{D}_q(\mathcal{D}_qf(z))}{\mathcal{D}_qf(z)}\right)$ lies in the region Ω_c bounded by the cardioid i.e.,

$$\psi_c(\mathcal{U}):=\{x+iy:(9x^2+9y^2-18x+5)^2-16(9x^2+9y^2-6x+1)=0\}.\text{ When}$$

$\mathcal{S}_{p,q}^*(\psi_c(z))$, it follows that $\frac{z\mathcal{D}_qf(z)}{[p]_qf(z)}\prec 1+(4/3)z+(2/3)z^2$ which yields $B_1=4/3$ and $B_2=2/3$. And therefore

$$|T_2(p+1)| \le \frac{4[p]_q^2}{81q^{2p}}\left(\frac{\left(8\frac{[p]_q}{q^p}+3\right)^2}{[2]_q^2}+36\right)\text{ and}$$

$$|T_3(p)| \le 1+\frac{32[p]_q^2}{9q^{2p}}+\frac{4[p]_q^2}{81[2]_q^2q^{2p}}\left(3+8\frac{[p]_q}{q^p}\right)\left(8\frac{[p]_q}{q^p}\big(2[2]_q-1\big)-3\right)\text{ for }\mathcal{S}_{p,q}(\psi_c(z)).$$

whereas

$$|T_2(p+1)| \le \frac{4[p]_q^4}{81q^{2p}}\left(\frac{\left(8\frac{[p]_q}{q^p}+3\right)^2}{[p+2]_q^2[2]_q^2}+\frac{36}{[p+1]^2}\right) \text{ and}$$

$$|T_3(p)| \le 1+\frac{32[p]_q^4}{9q^{2p}[p+1]_q^2} + \frac{4[p]_q^4}{81[2]_q^2q^{2p}[p+2]_q^2}\left(3+8\frac{[p]_q}{q^p}\right)\left[8\frac{[p]_q}{q^p}\left(\frac{2[p+2]_q[p]_q^2[2]_q}{[p+1]^2}-1\right)-3\right] \text{ for } \mathcal{C}_{p,q}(\psi_c(z))$$

$\boldsymbol{\psi(z)=\psi_{(\sin z)}:=1+\sin z, (z\in\mathcal{U})}$

By writing the Taylor series expansion for $\sin z$, we have $\psi(z):=\psi_{(\sin z)}=1+\sin z= 1+z-\frac{1}{6}z^3+\frac{1}{120}z^5+\cdots$. Thus, $B_1=1$ and $B_2=0$ which implies $|T_2(p+1)| \le \frac{[p]_q^2}{q^{2p}}\left(\frac{[p]_q^2}{[2]_q^2q^{2p}}+1\right)$ and $|T_3(p)| \le 1+\frac{[p]^2}{q^{2p}}\left[2+\frac{[p]_q^2}{[2]_q^2q^{2p}}(2[2]_q-1)\right]$ for $f\in\mathcal{S}_{p,q}^*(\psi_{\sin(z)})$. Similarly we can obtain $|T_2(p+1)| \le \frac{[p]_q^4}{q^{2p}}\left(\frac{[p]_q^2}{[p+2]_q^2[2]_q^2q^{2p}}+\frac{1}{[p+1]^2}\right)$ and $|T_3(p)| \le 1+\frac{[p]^4}{q^{2p}}\left[\frac{2}{[p+1]_q^2}+\frac{[p]_q^2}{[p+2]^2[2]_q^2q^{2p}}\left(\frac{2[2]_q[p]^2[p+2]_q}{[p+1]^2}-1\right)\right]$ for $f\in\mathcal{C}_{p,q}(\psi_{\sin z})$.

3.5 $\boldsymbol{\psi(z)=\psi_{\bigcirc}(z):=z+\sqrt{1+z^2}\ (z\in\mathcal{U})}$

Both $f\in\mathcal{S}_{p,q}(\psi)$ and $f\in\mathcal{C}_{p,q}(\psi_{\bigcirc})$consist of functions for which $\frac{z\mathcal{D}_qf(z)}{[p]_qf(z)}$ and $\frac{1}{[p]_q}\left(1+\frac{qz\mathcal{D}_q(\mathcal{D}_qf(z))}{\mathcal{D}_qf(z)}\right)$ lie in the leftmoon region $\Omega(\mathbb{C})$, defined as

$$\psi(\mathbb{C}(\mathcal{U}))=w\in\mathbb{C}: |w^2-1|<2|w|. \text{ Hence, we can conclude that}$$

$$\psi_{\bigcirc}(z):=z+\sqrt{1+z^2}=1+z+\frac{1}{2}z^2-\frac{1}{8}z^4+\cdots$$

Therefore, $B_1=1$ and $B_2=1/2$ which immediately yields

$$|T_2(p+1)| \le \frac{[p]_q^2}{4q^{2p}}\left[\frac{1}{[2]_q^2}\left(1+2\frac{[p]_q^2}{q^p}\right)^2+4\right] \text{ and}$$

$$|T_3(p)| \le 1+2\frac{[p]_q^2}{q^{2p}}+\frac{[p]_q^2}{4[2]_q^2q^{2p}}\left(1+2\frac{[p]_q}{q^p}\right)\left[2\frac{[p]_q}{q^p}(2[2]_q-1)-1\right] \text{ for } f\in\mathcal{S}_{p,q}^*(\psi).$$

Similarly, $f\in\mathcal{C}_{p,q}(\psi_{\bigcirc})$implies $\frac{1}{[p]_q}\left(1+\frac{qz\mathcal{D}_q(\mathcal{D}_qf(z))}{\mathcal{D}_qf(z)}\right)\prec z+\sqrt{1+z^2}$, and therefore we have $|T_2(p+1)| \le \frac{[p]_q^4}{q^{2p}}\left[\frac{1}{4[p+2]_q^2[2]_q^2}\left(1+2\frac{[p]_q^2}{q^p}\right)^2+\frac{1}{[p+1]_q^2}\right]$ and

$$|T_3(p)| \le 1+2\frac{[p]_q^4}{q^{2p}[p+1]^2} + \frac{[p]_q^4}{4[p+2]_q^2[2]_q^2q^{2p}}\left(1+2\frac{[p]_q}{q^p}\right)\left[2\frac{[p]_q}{q^p}\left(\frac{2[p]_q^2[p+2]_q[2]_q}{[p+1]_q^2}-1\right)-1\right].$$

$\boldsymbol{\psi(z)=\psi(z):=1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, (z\in\mathcal{U}).}$

The parabolic starlike class $\mathcal{S}_{p,q}^*(P)$ and the uniformly convex class $\mathcal{UCV}_{p,q}(P)$, derived from the Ma-Minda class of starlike and convex functions by replacing

$$\psi(z) := 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 = 1 + \frac{8}{\pi^2}z + \frac{16}{3\pi^2}z^2 + \frac{184}{45\pi^2}z^3 + \cdots$$

This yields $B_1 = 8/\pi^2$ and $B_2 = 16/3\pi^2$ and thus we get

$$|T_2(p+1)| \le \frac{[p]_q^2}{[2]_q^2 q^{2p}}\left[\frac{16}{3\pi^2} + \frac{64[p]_q}{q^p\pi^4}\right]^2 + \frac{16[p]_q^2}{q^{2p}\pi^4} \text{ and } |T_3(p)| \le 1 + \frac{128[p]_q^2}{q^{2p}\pi^4}$$
$$+ \frac{256[p]^2}{9q^{2p}[2]_q^2\pi^4}\left(1 + \frac{12[p]_q}{q^p\pi^2}\right)\left[\frac{12[p]_q\left(2[2]_q - 1\right)}{q^p\pi^2} - 1\right] \text{ for } f \in \mathcal{S}^*_{p,q}(P)$$

For $f \in \mathcal{UCV}_{p,q}(P)$, we get $|T_2(p+1)| \le \frac{256[p]_q^4}{9[p+2]_q^2[2]_q^2q^{2p}\pi^4}\left[1 + \frac{12[p]_q}{q^p\pi^2}\right]^2 + \frac{64[p]_q^4}{q^{2p}[p+1]_q^2\pi^4}$ and $|T_3(p)| \le 1 + \frac{128[p]_q^2}{q^{2p}[p+1]_q^2\pi^4} + \frac{256[p]_q^4}{9q^{2p}[p+2]_q^2[2]_q^2\pi^4}\left(1 + \frac{12[p]_q}{q^p\pi^2}\right)\left[\frac{12[p]_q}{q^p\pi^2}\left(\frac{2[p]_q^2[2]_q[p+2]_q}{[p+1]^2} - 1\right) - 1\right]$.

$\psi(z) = \psi_{(\lim*)}(z) := 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, (z \in \mathcal{U})$.

This $\psi(z)$ is associated with the limacon $(4u^2 + 4v^2 - 8u - 5)^2 + 8(4u^2 + 4v^2 - 12u - 3)$ and in this case, we have $B_1 = \sqrt{2}$ and $B_2 = \frac{1}{2}$. Hence, for $f \in \mathcal{S}^*_{p,q}(\lim(z))$, we obtain $|T_2(p+1)| \le \frac{[p]_q^2}{4q^{2p}}\left[\frac{1}{[2]_q^2}\left(1 + 4\frac{[p]_q^2}{q^p}\right)^2 + 8\right]$ and $|T_3(p)| \le 1 + 4\frac{[p]_q^2}{q^{2p}} + \frac{[p]_q^2}{4[2]_q^2q^{2p}}\left(1 + 4\frac{[p]_q}{q^p}\right)\left[4\frac{[p]_q}{q^p}\left(2[2]_q - 1\right) - 1\right]$. For $f \in \mathcal{C}_{p,q}(\lim(z))$, we have $|T_2(p+1)| \le \frac{[p]_q^4}{4q^{2p}}\left[\frac{1}{[p+2]_q^2[2]_q^2}\left(1 + 4\frac{[p]_q^2}{q^p}\right)^2 + \frac{8}{[p+1]_q^2}\right]$ and $|T_3(p)| \le 1 + 4\frac{[p]_q^4}{q^{2p}[1+p]^2} + \frac{[p]_q^4}{4[p+2]_q^2[2]_q^2q^{2p}}\left(1 + 4\frac{[p]_q}{q^p}\right)\left[4\frac{[p]_q}{q^p}\left(\frac{2[p]_q^2[p+2]_q[2]_q}{[p+1]_q^2} - 1\right) - 1\right]$.

$\psi(z) = \psi_p(Ne(z)) := 1 + z - \frac{z^3}{3}, (z \in \mathcal{U})$.

The function $\psi_{p,q}(Ne(z)) = 1 + z - \frac{z^3}{3}$ maps the unit disk into the interior of the 2-cusped kidney-shaped nephroid. In this case, we have $B_1 = 1$ and $B_2 = 0$, which leads to $|T_2(p+1)| \le \frac{[p]_q^2}{q^{2p}}\left(\frac{[p]_q^2}{[2]_q^2q^{2p}} + 1\right)$ and $|T_3(p)| \le 1 + \frac{[p]^2}{q^{2p}}\left[2 + \frac{[p]_q^2}{[2]_q^2q^{2p}}\left(2[2]_q - 1\right)\right]$ for $f \in \mathcal{S}^*_{p,q}(Ne(z))$. For $f \in \mathcal{C}_{p,q}(Ne(z))$, we obtain $|T_2(p+1)| \le \frac{[p]_q^4}{q^{2p}}\left(\frac{[p]_q^2}{[p+2]_q^2[2]_q^2q^{2p}} + \frac{1}{[p+1]^2}\right)$ and $|T_3(p)| \le 1 + \frac{[p]^4}{q^{2p}}\left[\frac{2}{[p+1]_q^2} + \frac{[p]_q^2}{[p+2]^2[2]_q^2q^{2p}}\left(\frac{2[2]_q[p]^2[p+2]_q}{[p+1]^2} - 1\right)\right]$.

3.9 $\psi(z) = \psi_{card}(z) := 1 + ze^z, (z \in \mathcal{U})$.

Both $\mathcal{S}^*_{p,q}(\text{card}(z)$ and $\mathcal{KC}$ consist of functions for which $\frac{z\mathcal{D}_qf(z)}{[p]_qf(z)}$ and $\frac{1}{[p]_q}\left(1 + \frac{qz\mathcal{D}_q\left(\mathcal{D}_qf(z)\right)}{\mathcal{D}_qf(z)}\right))$ map the unit disk onto a cardioid domain. We can derive the following estimates:

For $f \in \mathcal{S}^*_{p,q}\left(\text{card}(z): |T_2(p+1)| \le \frac{[p]_q^2}{q^{2p}}\left[\frac{1}{[2]_q^2}\left(1 + \frac{[p]_q}{q^p}\right)^2 + 1\right]\right.$, $|T_3(p)| \le 1 + \frac{2[p]_q^2}{q^{2p}} + \frac{[p]_q^2}{q^{2p}[2]_q^2}\left(1 + \frac{[p]_q}{q^p}\right)\left[\frac{[p]_q}{q^p}\left(2[2]_q - 1\right) - 1\right]$.

For $f \in \mathcal{KC}$: $|T_2(p+1)| \le \frac{[p]_q^4}{q^{2p}}\left[\frac{1}{[p+2]_q^2[2]_q^2}\left(1 + \frac{[p]_q}{q^p}\right)^2 + \frac{1}{[p+1]_q^2}\right]$ and $|T_3(p)| \le 1 + \frac{2[p]_q^4}{q^{2p}[p+1]_q^2} + \frac{[p]_q^4}{q^{2p}[2]_q^2[p+2]_q^2}\left(1 + \frac{[p]_q}{q^p}\right)\left[\frac{[p]_q}{q^p}\left(\frac{2[2]_q[p+2]_q[p]_q^2}{[p+1]_q^2} - 1\right) - 1\right]$.

$\psi(z) = \psi_{\mathbb{C}}(z) := \sqrt{1+z}, (z \in \mathcal{U})$.

Both $\mathcal{S}^*_{p,q}(\mathbb{C}(z))$ and $\mathcal{C}^*_{p,q}(\mathbb{C}(z))$ consist of functions for which $\frac{z\mathcal{D}_qf(z)}{[p]_qf(z)}$ and $\frac{1}{[p]_q}\left(1 + \frac{qz\mathcal{D}_q\left(\mathcal{D}_qf(z)\right)}{\mathcal{D}_qf(z)}\right)$ map the unit disk onto the right half of the lemniscate $|\omega^2 - 1| = 1$. We can derive the following estimates: For $\mathcal{S}^*_{p,q}(\mathbb{C}(z))$:

$$|T_2(p+1)| \leq \frac{[p]_q^2}{64q^{2p}}\left[\frac{1}{[2]_q^2}\left(\frac{2[p]_q}{q^p}-1\right)^2+16\right],$$
$$|T_3(p)| \leq 1+\frac{[p]_q^2}{2q^{2p}}+\frac{[p]_q^2}{64[2]_q^2q^{2p}}\left(\frac{2[p]_q}{q^p}-1\right)\left[\frac{2[p]_q}{q^p}\left(2[2]_q-1\right)-1\right].$$

For $\mathcal{C}^*_{p,q}(\mathbb{C}(z))$: $|T_2(p+1)| \leq \frac{[p]_q^4}{64q^{2p}}\left[\frac{1}{[p+2]_q^2[2]_q^2}\left(\frac{2[p]_q}{q^p}-1\right)^2+\frac{16}{[p+1]_q^2}\right]$, $|T_3(p)| \leq 1+\frac{[p]_q^4}{2q^{2p}[p+1]_q^2}+\frac{[p]_q^4}{64[2]_q^2q^{2p}[p+2]_q^2}\left(\frac{2[p]_q}{q^p}-1\right)\left[\frac{2[p]_q}{q^p}\left(\frac{2[2]_q[p]_q^2[p+2]_q}{[p+1]_q^2}-1\right)-1\right]$.

$\boldsymbol{\psi(z)=\psi_{\mathbb{C}}(z):=\frac{2}{1+e^{-z}},(z\in\mathcal{U}).}$

The modified sigmoid function $\psi_{\mathbb{C}}=\frac{2}{1+e^{-z}}$ maps the unit disk $\mathcal{U}$ onto the domain $\Delta_{SG}:=\omega\in\mathbb{C}:\left|\log\left(\frac{w}{2-w}\right)\right|<1$, which is symmetric with respect to the real axis. Moreover, the function $G(z)$ defined by $\psi_{\mathbb{C}}$ is convex and starlike with respect to $G(0)=1$.In addition, $G(z)$ has a positive real part in $\mathcal{U}$. Consequently, $G(z)$ belongs to the Ma-Minda class of functions. As a result the classes $S^*_{p,q}(G)$ and $C_{p,q}(G)$ naturally form subclasses of $\mathcal{S}^*_{p,q}(\psi)$ and $\mathcal{C}_{p,q}(\psi)$. We denote these subclasses by $\mathcal{S}^*_{p,q}(SG)$ and $\mathcal{C}_{p,q}(SG)$. Form an analytical viewpoint, a function $f\in\mathcal{S}^*_{p,q}(SG)$ if and only if $\frac{zD_qf(z)}{[p]_qf(z)}$ lies in the region ΔSG. Making use of the expression $\frac{2}{1+e^{-z}}=1+\frac{z}{2}-\frac{z^3}{12}\cdots$, we can deduce that, $B_1=\frac{1}{2}$, $B_2=0, |T_2(p+1)| \leq \frac{[p]_q^4}{16[2]_q^2q^{2p}}+\frac{[p]_q^2}{4q^{2p}}, |T_3(p)| \leq 1+\frac{[p]_q^2}{2q^{2p}}+\frac{[p]_q^4(2[2]_q-1)}{16q^{4p}[2]_q^2}$, and $|T_2(p+1)| \leq \frac{[p]_q^4}{4q^{2p}}\left[\frac{[p]_q^2}{4[p+2]_q^2[2]_q^2q^{2p}}+\frac{1}{[p+1]_q^2}\right], |T_3(p)| \leq 1+\frac{[p]_q^4}{2q^{2p}[1+p]_q^2}+\frac{[p]_q^6}{16q^{4p}[2]_q^2[p+2]_q^2}\left(\frac{2[2]_q[p]_q^2[p+2]_q}{[p+1]_q^2}-1\right)$, for our starlike and convex function classes.

Conclusion

In this paper, we studied Toeplitz determinants for q-starlike functions $\left(\mathcal{S}^*_{p,q}(\varphi)\right)$, q-convex functions $\left(\mathcal{C}_{p,q}(\varphi)\right)$, and q-bounded turning functions $\left(\mathcal{R}_{p,q}(\varphi)\right)$. We obtained four main results. Theorem 1 gives a sharp bound for $T_2(p+1)$ for q-starlike functions. Theorem 2 gives a sharp bound for $T_2(p+1)$ for q-convex functions. Theorem 3 gives a sharp bound for $T_3(p)$ for q-starlike functions. Theorem 4 gives a sharp bound for $T_3(p)$ for q-convex functions. All four bounds are sharp, meaning they cannot be improved. When $q\to1^-$, our results reduce to classical known results, confirming our work is a natural extension. We also found a duality between starlike and convex functions, where the convex bound is smaller than the starlike bound for large p. For practical use, our bounds apply to signal processing (worstcase prediction error for q-deformed filters), quantum information (maximum entanglement bounds), and finance (volatility persistence for risk management). Future work includes studying higher-order Toeplitz determinants, applying the same method to other q-classes, and testing these bounds with real data. In summary, our results are new, rigorous, and useful for applications in engineering, physics, and finance.

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