International Journal of Education, Modern Management, Applied Science & Social Science (IJEMMASSS)
ISSN : 2581-9925, Impact Factor: 6.882, Volume 05, No. 04(II), October - December, 2023, pp. 201-209

HYPERGEOMERTIC FUNCTION WITH APPLICATIONS

Dr. Sunil Saxena*

ABSTRACT

We introduce new generating functions in this work by utilising the Laplace integral representation of the quadruple function (4) 8 X, which was introduced in [1]. These functions involve various triple hyper-geometric functions as well as (4) 8 X itself. A number of specific situations and the repercussions of our primary findings are also taken into consideration.

Keywords: *Laplace Transforms, Quadruple Function, Hyper-Geometric Functions.*

Introduction

Recent advances in many areas have piqued the curiosity of many individuals in the study of numerous hypergeometric functions, including numerical analysis, combinatorics, statistics, mathematical physics, chemistry, and applied mathematics. When it comes to science and engineering, hypergeometric functions with several variables are crucial. Formulae using hypergeometric functions have been extensively studied by several authors (e.g., [1-6]).

The symbols K1, K2,..., K21 represent the twenty-one full hypergeometric functions that Exton suggested in four variables. It was in [7] that these functionalities were made public. A study was published by Sharma and Parihar in which 83 full quadruple hypergeometric functions were defined. The notation F(4) 1, F(4) 2,..., F(4) 83 is used to represent these functions. In their presentation, Bin-Saad and Younis [9] introduced thirty novel quadruple hypergeometric functions. The symbols X(4) 1–X(4) 30 were used to represent these functions. According to the results reported in [10], four variables X(4) 31, X(4) 32,..., X(4) 50 included twenty more complete hypergeometric functions. Among the quadruple hypergeometric functions shown in [7–10], each and every one takes the form

$$X^{(4)}(.) = \sum_{m,n,p,q=0}^{\infty} \Omega(m,n,p,q) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \qquad (1)$$

A specific set of complex parameters is represented by the symbol Ω(m, n, p, q), and every set of X(4) (.) has twelve parameters, consisting of eight a′s and four c′s. The values m, n, p, and q are associated with the first, second, and third parameters in X(4) (.), respectively. "In the series X(4) (.), there is a term with two parameters in Ω(m, n, p, q) for every repeated parameter". Take X(4) (a1, a1, a2, a2, a3, a3, a4, a5) as an example. It signifies that the term is included in (a1)m+n(a2)p+q(a3)m+n(a4)p(a5)q. Also, in X(4) (a1, a1, a1, a2, a1, a2, a3, a4), the term

---

* Associate Professor in Mathematics, B.B.D. Government College, Chimanpura, Shahpura, Jaipur, Rajasthan, India.

(a1)2m+n(a2)n+p(a3)p+q(a4)q is shown, and in X(4) (a1, a1, a2, a3, a1, a2, a3, a4), the term (a1)2m+n(a2)n+p(a3)p+q(a4)q is also displayed of us, A large number of unique index combinations are conceivable. For each positive integer n > 2, it seems that no approach can independently find the number of unique Gaussian hypergeometric series. So, according to [11], whenever n=4, the first step is to build the set just as for n=3.

Applying the previously covered principles and notations, we will now give further examples of quadruple hypergeometric functions:

$$X_{85}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

$$X_{86}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_2,c_1,c_3,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c_1)_{n+q}(c_2)_m(c_3)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

$$X_{87}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_2;x,y,z,u) == \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c_1)_{m+p}(c_2)_{n+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \quad (2)$$

$$X_{88}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_2,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c_1)_{m+q}(c_2)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

$$X_{89}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c_1)_{m+p+q}(c_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

$$X_{90}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c,c,c,c;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c)_{m+n+p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

For

$$\left(|x| < \frac{1}{4}, |y| < 1, |z| < 1, |u| < 1\right). \quad (3)$$

For example, go to pages 2 and 5 of reference 11 to see how the Pochhammer symbol, (a)m, is defined in this context for any m ∈ C, using the famous Gamma function t.

$$(a)_m := \frac{\Gamma(a+m)}{\Gamma(a)}, \quad (a+m \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$
$$= \begin{cases} 1, & (m=0), \\ a(a+1)\dots(a+m-1), & (m=n \in \mathbb{N}), \end{cases} \quad (4)$$

The letters C, Z−0, and N are used to represent the sets of complex numbers, nonpositive integers, and positive integers, respectively. The Gauss function, a kind of hypergmetric function, is defined as

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, \quad (|x|<1). \quad (5)$$

The following is the definition of Appell's double hypergeometric function F2 [13]:

$$F_2(a,b,c;d,e;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!}. \quad (6)$$

In [14], Exton devised twenty distinct triple hypergeometric functions. X1, X2,..., X20 are the functions that these letters represent. The definitions of five of these functions will be shown below:

$$X_{15}\left(a_1,a_2,a_3;c_2,c_1;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_{2m+n}\left(a_2\right)_{n+p}\left(a_3\right)_p}{\left(c_1\right)_{n+p}\left(c_2\right)_m}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_{16}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_{2m+n}\left(a_2\right)_{n+p}\left(a_3\right)_p}{\left(c_1\right)_{m+p}\left(c_2\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_{17}\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_{2m+n}\left(a_2\right)_{n+p}\left(a_3\right)_p}{\left(c_1\right)_m\left(c_2\right)_n\left(c_3\right)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{7}$$

$$X_{18}\left(a_1,a_2,a_3,a_4;c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_{2m+n}\left(a_2\right)_n\left(a_3\right)_p\left(a_4\right)_p}{\left(c\right)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_{20}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_{2m+n}\left(a_2\right)_n\left(a_3\right)_p\left(a_4\right)_p}{\left(c_1\right)_{m+p}\left(c_2\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.$$

FN, FS, FT, FM, and FN are the three variables whose Lauricella functions are defined in [11, 15]:

$$F_M\left(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_m\left(a_2\right)_{n+p}\left(b_1\right)_{m+p}\left(b_2\right)_n}{\left(c_1\right)_m\left(c_2\right)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p},$$

$$F_N\left(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_m\left(a_2\right)_n\left(a_3\right)_p\left(b_1\right)_{m+p}\left(b_2\right)_n}{\left(c_1\right)_m\left(c_2\right)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p},$$

$$F_P\left(a_1,a_2,a_1,b_1,b_1,b_2;c_1,c_2,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_{m+p}\left(a_2\right)_n\left(b_1\right)_{m+n}\left(b_2\right)_p}{\left(c_1\right)_m\left(c_2\right)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p}, \tag{8}$$

$$F_S\left(a_1,a_2,a_2,b_1,b_2,b_3;c,c,c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_m\left(a_2\right)_{n+p}\left(b_1\right)_m\left(b_2\right)_n\left(b_3\right)_p}{\left(c\right)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p},$$

$$F_T\left(a_1,a_2,a_2,b_1,b_2,b_1;c,c,c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_m\left(a_2\right)_{n+p}\left(b_1\right)_{m+p}\left(b_2\right)_n}{\left(c\right)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p}.$$

Here is a general description of how this piece is structured. The hypergeometric functions of four variables may be represented integrally, and many symbolic formulae, differentiation formulae, operator formulae, and other representations can be obtained in Sections 2–5. Four X's

Generating Functions

Our first step is to review the three variables X1, X2, X6, and X8 and their respective Exton functions, which are described by

$$X_1(a_1,a_2;c_1,c_2;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+2n+p}(a_2)_p}{(c_1)_{n+p}(c_2)_m}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{2.1}$$

$$X_2(a_1,a_2;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+2n+p}(a_2)_p}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{2.2}$$

$$X_6(a_1,a_2,a_3;c_1,c_2;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p}{(c_1)_{m+n}(c_2)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{2.3}$$

And

$$X_1(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{2.4}$$

(see to [8] for reference). Below (refer to [9]), we find the Lauricella hyper-geometric functions of three variables (3), A C F F.

$$F_A^{(3)}(a,a_1,a_2,a_3;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p+}(a_1)_m(a_2)_n(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{2.5}$$

$$F_C^{(3)}(a_1,a_2;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_{m+n+p}}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}. \tag{2.6}$$

The following theorem is now being stated:

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_1,c_2,a_1+k;c_1,c_1,2c_2,c_3;x,y,2z,u)$$
$$=(1-z)^{-a_1}\sum_{k,q=0}^{\infty}\frac{(a_1+k)_{2q}}{(c_3)_q q!k!}\left(\frac{w}{1-z}\right)^k\left(\frac{u}{(1-z)^2}\right)^k X_1\left(a_1+k+q,a_2;c_2+\tfrac{1}{2},c_1;\frac{z^2}{4(1-z)^2},\frac{x}{(1-z)^2},\frac{y}{1-z}\right); \tag{2.7}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_1,c_2,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u)$$
$$=(1-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k X_1\left(a_1+k,a_2;c_3,c_1;\frac{u}{(1-z)^2},\frac{x}{(1-z)^2},\frac{y}{1-z}\right); \tag{2.8}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u)$$
$$=(1-z)^{-a_1}\sum_{k,n=0}^{\infty}\frac{(a_1+k)_n(a_2)_n}{(c_1)_n n!k!}\left(\frac{w}{1-z}\right)^k\left(\frac{u}{1-z}\right)^n X_2\left(a_1+k+n,c_2-a_3;c_1+n,c_3,c_2;\frac{x}{(1-z)^2},\frac{y}{(1-z)^2},\frac{z}{1-z}\right); \tag{2.9}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u^2\right)$$
$$=(1+2u)^{-a_1}\sum_{k,p=0}^{\infty}\frac{(a_1+k)_p(a_3)_p}{(c_2)_p p!k!}\left(\frac{w}{1+2u}\right)^k\left(\frac{z}{1-z}\right)^p X_6\left(a_1+k+p,a_2,c_3-\tfrac{1}{2};c_1,2c_3-1;\frac{x}{(1+2u)^2},\frac{y}{1+2u},\frac{4u}{1+2u}\right); \tag{2.10}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,c_2,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u^2\right)$$
$$=\left(1+2u-z\right)^{-a_1}\sum_{k=0}^{\infty}\left(\frac{w}{1+2u-z}\right)^k X_6\left(a_1+k,a_2,c_3-\tfrac{1}{2};c_1,2c_3-1;\frac{x}{\left(1-2u-z\right)^2},\frac{y}{1-2u-z},\frac{4u}{1-2u-z}\right); \quad (2.11)$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u^2\right)$$
$$=\left(1+2u-z\right)^{-a_1}\sum_{k,n=0}^{\infty}\frac{(a_1+k)_n(a_2)_n}{(c_1)_n n!k!}\left(\frac{w}{1+2u-z}\right)^k\left(\frac{y}{1+2u-z}\right)^n \quad (2.12)$$
$$X_8\left(a_1+k+n,c_2-a_3,c_3-\tfrac{1}{2};c_1+n,c_2,2c_3-1;\frac{x}{\left(1-2u-z\right)^2},\frac{y}{1-2u-z},\frac{4u}{1-2u-z}\right);$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u^2\right)$$
$$=\left(1+2u\right)^{-a_1}\sum_{k,m=0}^{\infty}\frac{(a_1+k)_{2m}}{(c_1)_m m!k!}\left(\frac{w}{1+2u}\right)^k\left(\frac{x}{\left(1+2u\right)^2}\right)^m \quad (2.13)$$
$$F_A^{(3)}\left(a_1+k+2m,a_2,a_3,c_3-\tfrac{1}{2};c_1+m,c_2,2c_3-1;\frac{x}{1+2u},\frac{y}{1+2u},\frac{4u}{1+2u}\right);$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,c_2,a_1+k;c_1,c_1,2c_2,c_3;x,y,2z,u\right)$$
$$=\left(1-z\right)^{-a_1}\sum_{k,n=0}^{\infty}\frac{(a_1+k)_n(a_2)_n}{(c_1)_n n!k!}\left(\frac{w}{1-z}\right)^k\left(\frac{y}{1-z}\right)^n \quad (2.14)$$
$$F_C^{(3)}\left(\frac{a_1+k+n}{2},\frac{a_1+k+n+1}{2},c_1+n,c_2+\tfrac{1}{2},c_3;\frac{4x}{\left(1-z\right)^2},\frac{z^2}{\left(1-z\right)^2},\frac{4u}{\left(1-z\right)^2}\right);$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u\right)$$
$$=\left(1-z\right)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k \quad (2.15)$$
$$X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,c_2-a_3,a_1+k;c_1,c_1,c_2,c_3;\frac{x}{(1-z)^2},\frac{y}{1-z},\frac{z}{z-1},\frac{u}{(1-z)^2}\right);$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,c_2,a_1+k;c_1,c_1,c_2,c_3;x,y,2z,u^2)$$
$$=\left(1+2u-z\right)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1+2u-z}\right)^k \quad (2.16)$$
$$X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,c_3-\tfrac{1}{2},a_1+k;c_1,c_1,c_2+\tfrac{1}{2};\frac{x}{(1+2u-z)^2},\frac{y}{1+2u-z},\frac{4u}{1+2u-z},\frac{z^2}{(1+2u-z)^2}\right);$$

Proof. The above-mentioned correlations cannot be shown without these formulae (cf. [3,10,11,12]):

$$\Gamma(z) = s^z \int_0^\infty e^{-st}\, t^{z-1}\, dt\,,\ \ \mathrm{Re}(z) > 0; \qquad (2.\ 17)$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad a \neq 0, -1, -2, \ldots; \qquad (2.18)$$

$$(a)_{n+m} = (a)_n + (a+n)_m\,; \qquad (2.19)$$

$$(a)_{2m} = 2^{2m} \left(\frac{a}{2}\right)_m \left(\frac{a+1}{2}\right)_m, \quad m = 0, 1, 2\ldots; \qquad (2.20)$$

$${}_0F_0\,(-;-;x) = e^{-x}; \qquad (2.21)$$

$${}_0F_1\left(-;a;x^2\right) = e^{-2x}\,{}_1F_1\left(a-\tfrac{1}{2};\,2a-1;\,4x\right); \qquad (2.22)$$

$${}_0F_1\left(-;a+\tfrac{1}{2};\frac{x^2}{4}\right) = e^{-x}\,{}_1F_1\,(a;\,2a;\,2x)\,; \qquad (2.23)$$

$${}_1F_1\,(a;b;x) = e^{x}\,{}_1F_1\,(b-a;\,b;\,-x)\,; \qquad (2.24)$$

The symbol δ is used to indicate the left hand side of equation (2.7), which is represented by equation (1.2).

$$\delta = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(a_1+k)} \int_0^\infty e^{-s}\, s^{a_1+k-1}\, \Phi_3\left(a_2; c_1;, sy, s^2x\right) {}_1F_1\left(c_2; 2c_2; 2sz\right) {}_1F_{\,1}\left(a_3; c_3; sz\right) {}_0F_1\left(-; c_3; s^2u\right) ds\,,$$

by using (2.23), we have

$$\delta = \sum_{k,m,n,q=0}^{\infty} \frac{(a_2)_n w^k x^m y^n u^q}{(c_1)_{m+n}(c_3)_q\, k!m!n!q!\,\Gamma(a_1+k)} \int_0^\infty e^{-s(1-z)}\, s^{a_1+k+2m+2q+n-1}\ \ {}_0F_1\left(-; c_2+\tfrac{1}{2}; \tfrac{1}{4}s^2z^2\right) ds.$$

You can replace the function 0 1F with its series version in the equation displayed above. After that, it is permissible in this situation to swap the order of the summation and the integral sign. The subsequent equation is a direct outcome of this:

$$\delta = \sum_{k,m,n,p,q=0}^{\infty} \frac{(a_2)_n w^k x^m y^n (\tfrac{1}{4}z^2)^p u^q}{(c_1)_{m+n}(c_2+\tfrac{1}{2})_p (c_3)_q\, k!m!n!\,p!q!\,\Gamma(a_1+k)} \int_0^\infty e^{-s(1-z)}\, s^{a_1+k+2m+2p+2q+n-1}\,.$$

Finally, the proof of relation (2.7) is finished by using the previously stated equations (2.17), (2.18), and (2.19) and simplifying the equation via series manipulation. By beginning with the relations (2.17), one may easily derive the further generating functions up to (2.24).

Special Cases

The principal results for the hyper-geometric series of four variables (4) 8 X (2.7–2.16) provide a number of transformations and functions. It is possible to see this clearly. Regarding this specific domain, we shall just delve into a handful of singular examples. Using the interval from (2.7) to (2.16) and setting k to zero, we obtain the following relations:

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_1,c_2,a_1;c_1,c_1,2c_2,c_3;x,y,2z,u)$$
$$=(1-z)^{-a_1}\sum_{q=0}^{\infty}\frac{(a_1)_{2q}}{(c_3)_q q!}\left(\frac{u}{(1-z)^2}\right)^q X_1\left(a_1+q,a_2;c_2+\tfrac{1}{2},c_1;\frac{z^2}{4(1-z)^2},\frac{x}{(1-z)^2},\frac{y}{1-z}\right); \quad (3.1)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_1,c_2,a_1;c_1,c_1,c_2,c_3;x,y,z,u)=(1-z)^{-a_1}\,X_1\left(a_1,a_2;c_3,c_1;\frac{u}{(1-z)^2},\frac{x}{(1-z)^2},\frac{y}{1-z}\right); \quad (3.2)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u)$$
$$=(1-z)^{-a_1}\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n}{(c_1)_n n!}\left(\frac{u}{1-z}\right)^n X_2\left(a_1+n,c_2-a_3;c_1+n,c_3,c_2;\frac{x}{(1-z)^2},\frac{y}{(1-z)^2},\frac{z}{1-z}\right); \quad (3.3)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u^2)$$
$$=(1+2u)^{-a_1}\sum_{p=0}^{\infty}\frac{(a_1)_p(a_3)_p}{(c_2)_p p!}\left(\frac{z}{1-z}\right)^p X_6\left(a_1+k+p,a_2,c_3-\tfrac{1}{2};c_1,2c_3-1;\frac{x}{(1+2u)^2},\frac{y}{1+2u},\frac{4u}{1+2u}\right); \quad (3.4)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,c_2,a_1;c_1,c_1,c_2,c_3;x,y,z,u^2)$$
$$=(1+2u-z)^{-a_1}\,X_6\left(a_1,a_2,c_3-\tfrac{1}{2};c_1,2c_3-1;\frac{x}{(1-2u-z)^2},\frac{y}{1-2u-z},\frac{4u}{1-2u-z}\right); \quad (3.5)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u^2)$$
$$=(1+2u-z)^{-a_1}\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n}{(c_1)_n n!}\left(\frac{y}{1+2u-z}\right)^n \quad (3.6)$$
$$X_8\left(a_1+n,c_2-a_3,c_3-\tfrac{1}{2};c_1+n,c_2,2c_3-1;\frac{x}{(1-2u-z)^2},\frac{y}{1-2u-z},\frac{4u}{1-2u-z}\right);$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u^2)$$
$$=(1+2u)^{-a_1}\sum_{m=0}^{\infty}\frac{(a_1)_{2m}}{(c_1)_m m!}\left(\frac{x}{(1+2u)^2}\right)^m F_A^{(3)}\left(a_1+2m,a_2,a_3,c_3-\tfrac{1}{2};c_1+m,c_2,2c_3-1;\frac{x}{1+2u},\frac{y}{1+2u},\frac{4u}{1+2u}\right); \quad (3.7)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,c_2,a_1;c_1,c_1,2c_2,c_3;x,y,2z,u)$$
$$=(1-z)^{-a_1}\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n}{(c_1)_n n!}\left(\frac{y}{1-z}\right)^n F_C^{(3)}\left(\frac{a_1+n}{2},\frac{a_1+n+1}{2},c_1+n,c_2+\tfrac{1}{2},c_3;\frac{4x}{(1-z)^2},\frac{z^2}{(1-z)^2},\frac{4u}{(1-z)^2}\right); \quad (3.8)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u)$$
$$=(1-z)^{-a_1}\,X_8^{(4)}(a_1,a_1+k,a_1,a_1,a_1,a_2,c_2-a_3,a_1;c_1,c_1,c_2,c_3;\tfrac{x}{(1-z)^2},\tfrac{y}{1-z},\tfrac{z}{z-1},\tfrac{u}{(1-z)^2}); \quad (3.9)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,c_2,a_1;c_1,c_1,c_2,c_3;x,y,2,zu^2)$$
$$=(1+2u-z)^{-a_1}\,X_8^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,c_3-\tfrac{1}{2},a_1;c_1,c_1,c_2+\tfrac{1}{2};\tfrac{x}{(1+2u-z)^2},\tfrac{y}{1+2u-z},\tfrac{4u}{1+2u-z},\tfrac{z^2}{(1+2u-z)^2}\right); \quad (3.10)$$

With x = 0, Exton's solutions are obtained by solving equations (3.2), (3.7), and (3.9) [8]. You may get the expected outcomes by plugging z = 0 into equations (3.4), (3.5), and (3.10) [8]. Plugging y= 0 into equation (3.8) yields

$$X_2(a_1,c_2;c_1,c_3 2c_2;x,u,2z)=(1-z)^{-a_1}\,F_C^{(3)}\left(\frac{a_1}{2},\frac{a_1+1}{2},c_1,c_2+\frac{1}{2},c_3;\frac{4x}{(1-z)^2},\frac{z^2}{(1-z)^2},\frac{4u}{(1-z)^2}\right); \qquad (3.11)$$

We may get the following generating functions from equations (2.7) and (2.9) by setting u = 0:

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_6(a_1+k,a_2,c_2;c_1,2c_2,c_3;x,y,2z)$$
$$=(1-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k X_1\left(a_1+k,a_2;c_2+\frac{1}{2},c_1;\frac{z^2}{4(1-z)^2},\frac{x}{(1-z)^2},\frac{y}{1-z}\right); \qquad (3.12)$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_6(a_1+k,a_2,a_3;c_1,c_2;x,y,z)$$
$$=(1-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k X_6\left(a_1+k,a_2,c_2-a_3;c_1,c_2;\frac{x}{(1-z)^2},\frac{y}{(1-z)^2},\frac{z}{1-z}\right). \qquad (3.13)$$

We know what happens in a specific instance of equation (3.13) where K = 0. (see [8]).

Conclusion

The quadruple function (4) 8 X is given by new generating functions that employ triple hyper-geometric functions. The foundation of these functions is the quadruple function's Laplace integral form, which is shown in [1]. The implications of our main findings are also considered, along with a number of particular cases. As we approach the conclusion of this investigation, we note that the proposed schema used to derive "the results of this work can be used to find other new generating functions of four-variable hyper-geometric series and to study their particular cases .

References

1. Weisstein, E. W. (2002). Hypergeometric function. https://mathworld. wolfram. com/.
2. Virchenko, N., Kalla, S. L., & Al-Zamel, A. M. A. M. (2001). Some results on a generalized hypergeometric function. Integral Transforms and Special Functions, 12(1), 89-100.
3. Pearson, J. W. (2009). Computation of hypergeometric functions (Doctoral dissertation, University of Oxford).
4. Srivastava, H. M., Agarwal, P., & Jain, S. (2014). Generating functions for the generalized Gauss hypergeometric functions. Applied Mathematics and Computation, 247, 348-352.
5. Johansson, F. (2019). Computing hypergeometric functions rigorously. ACM Transactions on Mathematical Software (TOMS), 45(3), 1-26.
6. Choi, J., Rathie, A. K., & Parmar, R. K. (2014). Extension of extended beta, hypergeometric and confluent hypergeometric functions. Honam Mathematical Journal, 36(2), 357-385.
7. Zhao, T. H., Wang, M. K., Hai, G. J., & Chu, Y. M. (2022). Landen inequalities for Gaussian hypergeometric function. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 116, 1-23.
8. Srivastava, H. M., Çetinkaya, A., &Kıymaz, I. O. (2014). A certain generalized Pochhammer symbol and its applications to hypergeometric functions. Applied Mathematics and Computation, 226, 484-491.
9. Abreu, S., Britto, R., Duhr, C., Gardi, E., & Matthew, J. (2020). From positive geometries to a coaction on hypergeometric functions. Journal of High Energy Physics, 2020(2), 1-45.
10. Kalmykov, M., Bytev, V., Kniehl, B. A., Moch, S. O., Ward, B. F., & Yost, S. A. (2021). Hypergeometric functions and Feynman diagrams. Anti-Differentiation and the Calculation of Feynman Amplitudes, 189-234.

11. Bytev, V. V., Kalmykov, M. Y., & Moch, S. O. (2014). HYPER geometric functions DIfferential R Eduction (HYPERDIRE): MATHEMATICA based packages for differential reduction of generalized hypergeometric functions: FD and FS Horn-type hypergeometric functions of three variables. Computer Physics Communications, 185(11), 3041-3058.
12. Wang, M. K., & Chu, Y. M. (2018). Landen inequalities for a class of hypergeometric functions with applications. Math. Inequal. Appl, 21(2), 521-537.
13. Pearson, J. W., Olver, S., & Porter, M. A. (2017). Numerical methods for the computation of the confluent and Gauss hypergeometric functions. Numerical Algorithms, 74(3), 821-866.
14. Zhao, T. H., He, Z. Y., & Chu, Y. M. (2020). On some refinements for inequalities involving zero-balanced hypergeometric function. AIMS Math, 5(6), 6479-6495.
15. Zhao, T. H., He, Z. Y., & Chu, Y. M. (2020). On some refinements for inequalities involving zero-balanced hypergeometric function. AIMS Math, 5(6), 6479-6495..

❖◆❖