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CONTROLLER DESIGN BY USING MODEL ORDER REDUCTION TECHNIQUES

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ABSTRACT

Modeling of physical systems usually results in system of higher order generally greater than two. Design of controllers for such physical system becomes tedious when system order is high. So, it is desirable to approximate these models by reduced order models. In this paper a reduced order model is obtain by Modified Cauer Form and factor division algorithm. The controller is designed for the reduced order model and is connected in cascade with the original system to obtain the desired specifications. The proposed method assures the stability of the system under the reduced order model case. The proposed method is illustrated by a numerical example.

Keywords: *Modified Cauer Form, Order Reduction, Factor Division, Transfer function, PID Controller.*

Introduction

The quality of a reduced order model is judged by the way it is utilized, and the degree of its success in representing the desired characteristics of the system. One of the main objectives of order reduction is to design a controller of low order which can effectively con effectively control the original system high order system so that the overall system is of low order and is easy to understand. It is thus important that the model order reduction methods should reduce the high order controller to a low order controller without incurring too much error. Model reduction is based on

open loop considerations while closed loop stability performance is of main concern in controller design. Pade approximation [1,2] is the method of model order reduction of the higher order system. This gives the simplification of a model after converting it into a reduced order model. A different approach can be used to simplify a model which results are stable model. In this approach the numerator coefficients can be obtained by factor division method [3,4] and denominator coefficients can be obtained by modified cauer form[5,6]. Several methods [7, 8, 9, 10 and 11] have been developed for designing a PID controller.

In this paper a simple algebraic scheme is proposed to design a PID controller for Linear Time Invariant Continuous System. The closed loop transfer function of the reduced order models with PID controller are compared with the reference model transfer function in frequency domain.

Statement of the Problem

- **PID Controller Transfer Function**

PID controller can be mathematically represented as[12],

$$u(t) = k[e(t) + \frac{1}{T_i}\int_0^t e(\tau)dt + T_d \frac{de(\tau)}{dt} \quad (1)$$

Where $u(t)$ and $e(t)$ denotes the control and error signals of the system. is the proportion gain, and represents the integral and derivative time constants respectively. The corresponding PID controller transfer function is given as

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$G_c(s) = k[1 + \frac{1}{T_i s} + T_d s]$(2)

Equation (2) can be rewritten as

$G_c(s) = k_1 + \frac{k_2}{s} + k_3 s$(3)

k_1, k_2 and k_3 are represents the proportional, integral and derivative gain values of the controller.

- **Higher Order Transfer Function**

Let higher order system or process whose performance is unsatisfactory may be described by the transfer function

$G_n(s) = \frac{N(s)}{D(s)} = \frac{a_0 + a_1 s + a_2 s^2 + \cdots a_{n-1} s^{n-1}}{b_0 + b_1 s + b_2 s^2 + \cdots . b_n s^n}$(4)

and a reference model having the desired performance is given.

- **Lower Order Transfer Function**

To find a k^{th} lower order model for the above continuous system, where $k < n$ in the following form, such that the lower order model retains the characteristics of the original system and approximates its response as closely as possible for the same type of inputs.

$R_k(s) = \frac{N(s)}{D(s)} = \frac{c_0 + c_1 s + c_2 s^2 + \cdots c_{k-1} s^{k-1}}{d_0 + d_1 s + d_2 s^2 + \cdots . d_k s^k}$(5)

Where, cj and d_i are scalar constants.

Objective is to derive a controller such that the performance of the augmented process matches with that of the reference model. To reduce the computational complexities and difficulties of implementation, the higher order of the system is reduced into lower second order system. And PID controller is also derived for reduced order system.

Reduction Method

The reduction procedure for getting the k^{th}-order reduced models consists of the following two steps:

- **Step 1**

Determination of the denominator polynomial $D_m(s)$ for the k^{th}order reduced model by using modified Cauer form [8]: The brief procedure for getting $D_k(s)$using modified Cauer form is as follows:

- The given higher order system $G_n(s)$
- Without loss of generality, the coefficient of highest power of s^n equations (1) & (2) can always made unity and numerator is one degree lower than the denominator.

$G_n(s)$ can be expanded into a Cauer type continued fraction about s=0 and s=∞ alternately as follows:

$$G_n(s) = \cfrac{1}{\cfrac{a11}{b11} + \cfrac{s}{\cfrac{b1n}{b2n} + \cfrac{1}{\cfrac{a21}{b21} + \cdots \cfrac{s}{\cfrac{b2,n-1}{a3,n-1} + \cfrac{1}{\cdots}}}}}$$

Where

$a_{21} = a_{12} - \frac{a_{11} b_{12}}{b_{11}}$; $b_{21} = b_{11} - \frac{b_{1n} a_{21}}{a_{2n}}$

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$a_{2,n-1} = a_{1,n} - \frac{a_{11} b_{1n}}{b_{11}}$, $b_{2,n-1} = b_{1,n-1} - \frac{b_{1n} a_{2,n-1}}{a_{2,n}}$

$a_{2,n} = 1$

By continuing the above sequence of expansion we get the following form

$$G_n(s) \cfrac{1}{h_1 + \cfrac{s}{H_1 + \cfrac{1}{h_2 + \cfrac{s}{H_2 + \cfrac{1}{\cdots}}}}} \quad (3)$$

Where the quotients h_1, h_2,………….H_2,H_1……..are evaluated from the following modified array

$$\left.\begin{array}{lllll}
a_{11} & a_{12} & a_{1,n-1} & a_{1,n} & 1 \\
b_{11} & b_{12} & b_{1,n-1} & b_{1,n} & \\
a_{21} & a_{22} & a_{2,n-1} & 1 & \\
b_{21} & b_{22} & b_{2,n-1} & & \\
a_{31} & a_{32} & 1 & & \\
\ldots\ldots & & & & \\
\ldots\ldots & & & & \\
b_{n-1,1} & b_{n-1,2} & & & \\
a_{n,1} & & & & \\
b_{n,1} & & & & \\
1 & & & & \\
\multicolumn{5}{l}{h_1=\frac{a_{11}}{b_{11}},\quad h_2=\frac{a_{21}}{b_{21}}\ ,\ldots\ldots\ldots.h_n=\frac{a_{n,1}}{b_{n,1}}}
\end{array}\right\}$$

$$H_1 = b_{1,n} \qquad (4)$$

$$H_2 = b_{2,n-1}$$

$$H_{n-1} = b_{n-1,2}$$

$$H_n = b_{n,1}$$

Here the coefficients in equation (1) form the first two rows and remaining elements starting with third row are obtained from the recursive relations.

$$\left.\begin{array}{l}
a_{j+1,k}=a_{j,k+1}-h_j b_{j,k+1} \\
\text{and} \\
b_{j+1,k}=b_{j,k}-H_j a_{j+1,k} \quad j=1,2,\ldots\ldots\ldots n-1 \\
\text{where} \quad h_j=\frac{a_{j,1}}{b_{j,1}} \qquad (5) \\
H_j=\frac{b_{j,n+1-j}}{a_{j+1,n+1-j}} j=1,2,\ldots\ldots\ldots\ldots\ldots..n
\end{array}\right\}$$

It is to be noted that the end elements of all the odd rows can be written directly as

$a_{1,n+1}=a_{2,n}=a_{3,n-1}=$……………………$=a_{n+1,1}=1$

A reduced model of order m, is obtained by truncating equation (3) after the first 2m terms and inverting it to yield the transfer function formed out of the values of the quotients h_j and H_j derived in equation (4) is done by constructing the inversion table as follows

$$\left.\begin{array}{llll}
a_{11} & & & \\
b_{11} & & & \\
a_{21} & 1 & & \\
b_{21} & b_{22} & & \\
\ldots\ldots & & & \\
 & \ldots\ldots & & \\
a_{m,1} & a_{m,2}\ldots\ldots a_{m,m-1} & 1 & \\
b_{m,1} & b_{m,2}\ldots\ldots b_{m,m-1} & b_{m,m} & (6) \\
a_{m+1,1} & a_{m+1,2}\ldots\ldots a_{m+1,m} & 1 &
\end{array}\right\}$$

$$H_m=\frac{a_{11}}{b_{11}}, h_m=\frac{b_{11}}{a_{21}}$$

$$H_{m-1}=\frac{1}{b_{22}}\ , h_{m-1}=\frac{b_{21}}{a_{31}}$$

…………………………………..

…………………………………..

$H_1 = \frac{1}{b_{m,m}} h_1 = \frac{b_{m,1}}{a_{m+1,1}}$

Starting with $a_{11}=1$, the elements of the 2nd, third and subsequent rows are evaluated recursively from.

$a_{j+1,1}= h_{m+1-j}.b_{j,1}$ j=1,2,……………………m

$a_{j,k}= a_{j-1,k-1}+h_{m+2-j}\, b_{j-1,k}$ k=2,3,……………………j-1

and $b_{j,k}=b_{j-1,k}+H_{m+1-j}\, a_{j,k}$ j=2,3…………….m

k=1, 2……………j-1

It is also evident that the end elements can be written by inspection according as

$$\left.\begin{array}{ll} a_{jj}=1 & j=1,2,\ldots\ldots m+1 \\ b_{jj}=H_{m+1-j} & j=1,2,\ldots\ldots m \end{array}\right\} \quad (7)$$

There must be (2k+1) rows in complete array. The required denominator of the reduced order model equation (2) can be written from (2k+1)th row of the array as

$D_k(s)=s^k+a_{k+1,k}s^{k-1}+\ldots\ldots a_{k+1,2}s+a_{k+1,1}$

- **Step 2**

Determination of the numerator of kth order reduced model using Factor Division algorithm [11]

After obtaining the reduced denominator, the numerator of the reduced model is determined as follows:

$$N_k(s) = \frac{N(s)}{D(s)} \times D_{k(s)} = \frac{N(s)}{D(s)/D_{k(s)}} \quad (8)$$

Where $D_k(s)$ is reduced order denominator

There are two approaches for determining of numerator of reduced order model.

By performing the product of N(s) and $D_k(s)$ as the first row of factor division algorithm and D(s) as the second row up to s^{k-1} terms are needed in both rows.

By expressing $N(s)D_k(s)/D(s)$ as $N(s)/[D(s)/D_k(s)]$ and using factor division algorithm twice; the first time to find the term up to s^{k-1} in the expansion of $D(s)/D_k(s)$(i.e. put D(s) in the first row and $D_k(s)$ in the second row, using only terms up to s^{k-1}), and second time with N(s) in the first row and the expansion $[D(s)/D_k(s)]$ in the second row.

Therefore the numerator $N_k(s)$ of the reduced order model ($R_k(s)$) in eq.(4) will be the series expansion of

$$\frac{N(s)}{\frac{D(s)}{D_{k(s)}}} = \frac{\sum_{i=0}^{k-1} c_i s^i}{\sum_{i=0}^{k} d_i s^i} \quad (9)$$

About s=0 up to term of order s^{k-1}.

This is easily obtained by modifing the moment generating[14].which uses the familiar routh recurrence formulae to generate the third, fifth, and seventh etc rows as,

$$\propto_0 = \frac{g_0}{h_0} < \begin{matrix} g_0 & g_1 & g_2. & \ldots\ldots & g_{k-1} \\ h_0 & h_1 & h_2. & \ldots\ldots & h_{k-1} \end{matrix}$$

$$\propto_1 = \frac{l_0}{h_0} < \begin{matrix} l_0 & l_1 & l_2 & \ldots\ldots & l_{k-2} \\ h_0 & h_1 & h_2 & \ldots\ldots & h_{k-2} \end{matrix}$$

$$\propto_2 = \frac{m_0}{h_0} < \begin{matrix} m_0 & m_1 & m_2 & \ldots\ldots & m_{k-3} \\ h_0 & h_1 & h_2 & \ldots\ldots & h_{k-3} \end{matrix}$$

…………………………………………

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$$\propto_{k-2} = \frac{p_0}{h_0} < \begin{matrix} p_0 & p_1 \\ h_0 & h_1 \end{matrix}$$

$$\propto_{k-1} = \frac{q_0}{h_0} < \begin{matrix} q_0 \\ h_0 \end{matrix}$$

Where

$l_i = g_{i+1} - \propto_0 * h_{i+1}$,i=0,1,2,..........................

$m_i = l_{i+1} - \propto_1 * h_{i+1}$,i=0,1,2......................

....

....

$p_0 = p_1 - \propto_{k-2} h_1$

Therefore, the numerator $N_k(s)$of eq.(2) is given by

$N_k(s)=\sum_{i=0}^{k-1} \propto_i s^i$(10)

General Algorithm for Designing the PID Controller

- **Step 1**

Construction of a reference model M(s) on the basis of specifications whose closed loop system must approximate to that of the original closed loop response. Let it be specified as:

$M(s)=\frac{a_0+a_1 s+\cdots.........+.a_m s^m}{b_0+b_1 s+\cdots............+b_n s^n}$(11)

- **Step 2**

Determine an equivalent open loop specification model $\widetilde{M(s)}$

$\widetilde{M(s)} = \frac{M(s)}{1-M(s)}$(12)

- **Step 3**

Specified the structure of the controller

Let the controller structure is given by

$R_c(s)=\frac{p_0+p_1 s+\cdots..........+p_k s^k}{q_0+q_1 s+\cdots......+q_1 s^l}$ (13)

- **Step 4**

For determining the unknown controller parameters, the response of the closed loop system is matched with reference model as

$R_c(s)R_p(s) = \widetilde{M(s)}$

$R_c(s) = \frac{\widetilde{M(s)}}{R_c(s)} = \sum_{i=0}^{\infty} e_i s^i$ (14)

Where e_i are the power series expansion coefficients about s=0.

- **Step 5**

Now the unknown control parameters p_i and q_i are obtained by equating the equation (13) and (14) in Pade sense

$p_0=q_0e_0$

$p_1 = q_0 e_1 + q_1 e_0$

$p_2 = q_0 e_2 + q_1 e_1 + q_2 e_0$

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$p_i = q_0 e_i + q_1 e_{i-1} + \cdots . q_i e_0.$

$0 = q_0 e_{i+1} + q_1 e_i + \cdots . q_{i+1} e_0$

.

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$0 = q_0 e_{i+j} + q_1 e_{i+j-1} + \cdots . q_j e_0$

The controller having the desired structure is obtained by solving above linear equations

- **Step 6**

After obtaining the controller parameters, the close loop transfer function can be obtained as

$$\boldsymbol{R_{cl}} = \frac{\boldsymbol{R_c(s)R_k(s)}}{\boldsymbol{1 + R_c(s)R_k(s)}}$$

Numerical Example

The proposed method explains by considering numerical example, taken from the literature.

Example: Consider sixth order system

$$G(s) = \frac{72 + 54s + 12s^2 + s^3}{100 + 180s + 97s^2 + 18s^3 + s^4}$$

- **Step 1**

The denominator of reduced order model R_m(s) can be evaluate using equation for m=2. Make modified Routh Array & evaluate the quotientsh_1, h_2,h_3 and $H_{1,}H_2$,H_3.

Modified Routh Array

100	180	97	18	1
72	54	12	1	
105.048	80.344	16.612	1	
–33.048	–26.344	–4.612		
–3.377	1.955	1		
–48.622	–17.327			
3.150	1			
5.958				
1				

$h_{1\,=} \dfrac{100}{72} = 1.388,$ $\quad H_1 = \dfrac{1}{1} = 1$

$h_2 = \dfrac{105.048}{-33.048} = -3.178$ $H_2 = -\dfrac{4.612}{1} = -4.612$

$h_{3=} \dfrac{-3.377}{-48.622} = 0.069$ $\quad , H_3 = \dfrac{-17.327}{1} = -17.327$

$h_{4\,=} \dfrac{3.150}{5.958} = 0.528$ $\quad , H_4 = \dfrac{5.958}{1} = 5.958$

With the knowledge of the first four quotient (m=2), $h_{1\,=}$ 1.388, $H_{1=}$ 1, $h_{2=} - 3.178$, $H_{2=} - 4.612$ and with the help of equation construct the inversion table as follows

Inversion Table

1		
–4.612		
14.656	1	
10.044	1	
13.941	16.044	1

Hence the denominator D_2(s) of the reduced order model R_2(s)is obtained as:

D_2(s) =$s^2 + 16.044s + 13.941$

- **Step 2**

Now using the factor division method the numerator of reduced order model is given as

Consider $D_4(s)/D_2\,(s)$ given

$\propto_0 = 1.173 \;<\; \begin{matrix} 100 & 180 \\ 13.941 & 16.044 \end{matrix}$

$\propto_1 = 4.656 \;<\; \begin{matrix} 64.916 \\ 13.941 \end{matrix}$

$D_4(s)/D_2\,(s) = 7.173 + 4.656s$

Now considering N_4(s)/D_4 (s)/D_2(s)

$\propto_0 = 10.037 \ < \begin{matrix} 72 & 54 \\ 7.173 & 4.656 \end{matrix}$

$\propto_1 = 1.013 \ < \begin{matrix} 7.267 \\ 7.173 \end{matrix}$

Thus, Reduced Numerator is given as N_2 (s) =10.037+1.013s

Thus, the Reduced model is given as:

$$R_2\,(s) = \frac{10.037 + 1.013s}{13.941 + 16.044s + s^2}$$

PID Controller Design Using Reduced Order Model

Consider a reference model

M(s)=$\frac{4.242s+25}{s^2+7.07s+25}$

The equivalent open loop transfer function is:

$$\widetilde{M(s)} = \frac{4.242s + 25}{s^2 + 2.828s}$$

The reduced controller transfer function is:

$$R_c(s) = \frac{\widetilde{M(s)}}{R_2(s)}$$

$$= \frac{12.278 + 10.634s - 1.992s^2}{s}$$

It is compared by PID Controller transfer function.

$$= K_1 + \frac{K_2}{S} + K_3\,(s)$$

And the value of controller parameters are obtained as:

$K_1 = 10.634$, $K_2 = 12.278$,$K_3 = -1.992$

Thus, corresponding closed loop transfer function is:

$$R_{CL}(s) = \frac{R_c(s) * R_2(s)}{1 + R_c(s) * R_2(s)}$$

$$= \frac{5.223s^2 + 22.38s + 16.2}{s^3 + 9.214s^2 + 23.38s + 16.2}$$

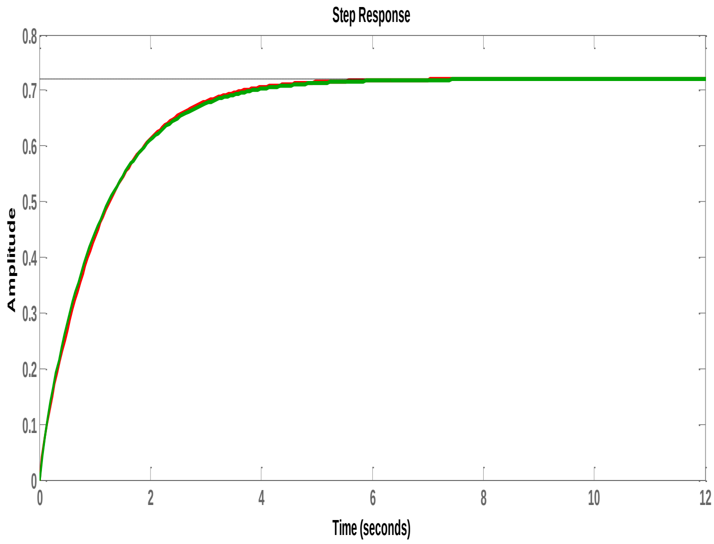

Fig.1: Comparison between Step Response of Original Plant and Reduced Order System

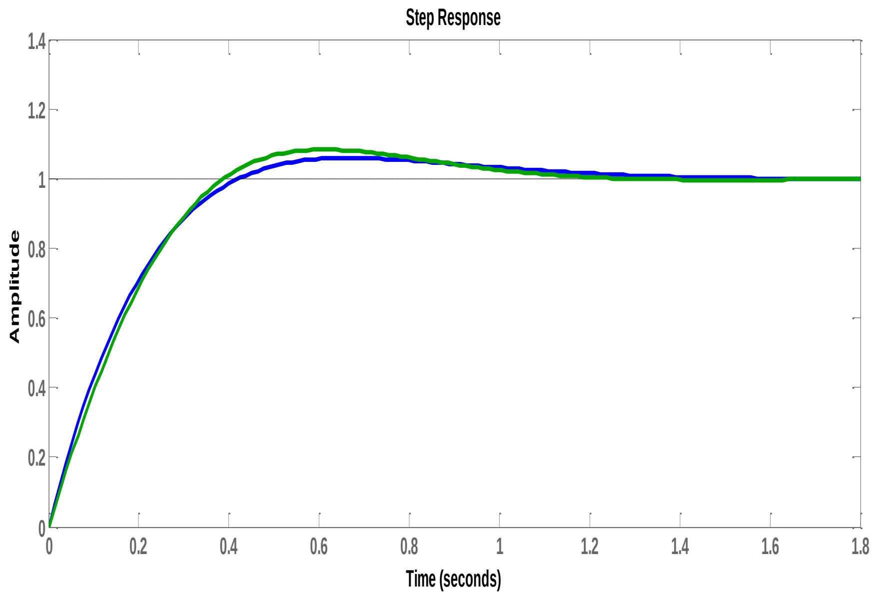

Fig. 2: Comparison between Step Response of Reference Model and Close Loop System

Table-I: A comparison of different parameters of original plant(G(s)), Reduced Order Plant(R_2(s),and closed loop system(R_{cl}(s))

Sr. No.	Parameter	G(s)	R_2(s)	R_{cl}(s)
1	Rise Time(Sec)	2.3149	2.2600	0.3015
2	Settling Time(Sec)	4.0227	4.0022	1.13
3	Peak Time(Sec)	7.7084	7.82	0.70
4	Overshoot	0	0	6.02
5	Peak	0.7196	0.74	1.02

Conclusion

In this proposed order reduction method for the linear single-input-single-output high order systems, the determination of denominator polynomial of the reduced model is done by using the Modified Cauer Form while the numerator coefficients are computed by factor division method. After that PID controller has been design for reduced order model. The proposed algorithm has been explained with a numerical example. A comparison of the step responses of the original and reduced system of second order are shown in the Figure 1 and a comparison of the step responses of the reference model and close loop reduced order system after including PID controller in loop is shown in figure 2. Table-I gives the comparison on the basis of various parameters of original plant and second order reduced model and closed loop responses. This method is simple, proficient and takes little computational time.

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