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LINEAR STABILITY OF LIBRATION POINTS IN THE RESTRICTED THREE-BODY PROBLEM WITH VARIABLE MASS WHEN THE SMALLER PRIMARY IS AN OBLATE SPHEROID

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ABSTRACT

The paper deals with the linear stability of the libration points in the restricted three body problem with variable mass when the smaller primary is an oblate spheroid. With the help of roots of the characteristic equation the nature of stability of libration points have been established. It is found that all the collinear libration points are unstable and triangular libration points are stable in linear sense.

Keywords: *Restricted Three-Body Problem, Jean's law, Space-Time-Transformation, Libration Points, Linear Stability*.

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Introduction

Shrivastava and Ishwar (1983) developed the equations of motion of the infinitesimal body of slowly reducing mass with time in the restricted three – body problem with the help of Jeans Law (1928) and Meshcherskii's (1949) space time transformation. Singh et al.(1984) studied the effect of perturbation on the location of libration points in the restricted three – body problem with variable mass and further (1985) he extended his work by discussing the effect of perturbation on the stability of the triangular libration points in the restricted three – body problem with variable mass. Das et al. (1988) established the equation of the infinitesimal body of slowly reducing mass with time in the elliptic restricted three – body problem.

Lukyanov (1990) showed that all types of libration points are stable under the conditions satisfying Meshcherskii's space time transformation. In the same year, El – Shaboury established the equations of motion of the infinitesimal body of reducing mass with time in the gravitational field of the two triaxial primaries with the help of Jeans Law (1928), Meshcherskii's space time transformation (1949) and Nechvili's transformation (1926). Plastino et al. (1992) developed a technique to show the velocity of Newton's second law of motion on the restricted three – body problem with variable mass and found that Newton's second law of motion is invalid for the problem with variable mass. Bekov (1993a, b) discussed the existence and stability of libration points in the R3BP and characterized the Hill's surface. Singh (2008) has also discussed the non – linear stability of libration points in the CR3BP with variable mass and further in 2010 extended his work in 2008 by taking the bigger primary as an oblate spheroid. In both the works, he used the analytical process developed by earlier authors.

In the present work, we have proposed to study the linear stability of libration points by using the nature of roots λ_1^2 and λ_2^2 of the characteristic equation corresponding to different libration points L_i(i=1,2,3,4,5)

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Equations of Motion

Consideringa bary-centric co-ordinate system (O, xyz), rotating relative to inertial frame the angular velocity μ . Taking the line joining the centers of the primaries as the $x-$axis, a line lying on the plane of motion and perpendicular to the $x-$axis andthrough the centre of mass as the $y-$axis, a line through the centre of mass and perpendicular to the plane of motion as the $z-$axis. Let(x, y, z) be the co-ordinates of the infinitesimal body with variable mass then the equation of motion of the infinitesimal body in inertial frame under the gravitational field of two primaries at $P_1(«, 0, 0)$ and $P_2(« -1, 0, 0)$can be written as

$$\frac{d}{dt}\left(m\frac{d\vec{r}}{dt}\right) = -Gm\left[\frac{1-\mu}{\rho_1^3}\vec{\rho}_1 + \frac{\mu}{\rho_2^3}\vec{\rho}_2 + \frac{9I\mu}{2\rho_2^5}\vec{\rho}_2\right] \text{..........(1)}$$

where $°_1^2 = (x-«)^2 + y^2 + z^2$ and $°_2^2 = (x-«+1)^2 + y^2 + z^2$.

$$\frac{d\vec{r}}{dt} = \frac{\partial\vec{r}}{\partial t} + \vec{\omega}\times\vec{r}\ . \qquad \text{.............(2)}$$

The oblateness parameter of the smaller primary is given by

$$I = \frac{a^2 - c^2}{5R^2} = \frac{R_e^{\ 2} - R_p^{\ 2}}{5R^2}, \quad \mu^2 = 1 + \frac{3}{2}I,$$

where $a = R_e$ and $c = R_p$ are the equatorial and polar radii of the oblate primary, $P_1P_2 = R$ is the dimensional distance between the primaries,

As in Hassan and Sweta(2017), the equation of motion of the infinitesimal body in rotating frame can be written as

$$\left(\frac{\partial}{\partial t} + \vec{\omega}\times\right)\left(m\frac{\partial\vec{r}}{\partial t} + m\vec{\omega}\times\vec{r}\right) = -Gm\left[\frac{1-\mu}{\rho_1^3}\vec{\rho}_1 + \frac{\mu}{\rho_2^3}\vec{\rho}_2 + \frac{9I\mu}{2\rho_2^5}\vec{\rho}_2\right],$$

$$\left.\begin{aligned} &\Rightarrow m\ddot{\vec{r}} + \dot{m}\left(\dot{\vec{r}} + \vec{\omega}\times\vec{r}\right) + 2m\vec{\omega}\times\vec{r} = \omega^2 m\left(x\hat{i} + y\hat{j}\right) - Gm\left[\frac{1-\mu}{\rho_1^3}\vec{\rho}_1 + \frac{\mu}{\rho_2^3}\vec{\rho}_2 + \frac{9I\mu}{2\rho_2^5}\vec{\rho}_2\right] \\ &= m\left[\omega^2 x\hat{i} + \omega^2 y\hat{j} - \frac{(1-\mu)(x-\mu)\hat{i} + y\hat{j} + z\hat{k}}{\rho_1^3} - \frac{\mu}{\rho_2^3}\left\{(x-\mu+1)\hat{i} + y\hat{j} + z\hat{k}\right\}\right. \\ &\left. - 9\frac{I\mu}{2\rho_2^5}\left\{(x-\mu+1)\hat{i} + y\hat{j} + z\hat{k}\right\}\right], \end{aligned}\right\} \text{..(3)}$$

where units are so chosen that the sum of the masses of the primaries and the gravitational constant G both are unity.

Thus, the equations of motion given by Hassan et al. (2017)in cartesian form are

$$\left.\begin{aligned} \ddot{x} + \frac{\dot{m}}{m}(\dot{x} - \omega y) + 2\omega\dot{x} &= -\frac{1}{m}\Omega_x, \\ \ddot{y} + \frac{\dot{m}}{m}(\dot{y} + \omega x) - 2\omega\dot{y} &= -\frac{1}{m}\Omega_y, \\ \ddot{z} + \frac{\dot{m}}{m}\dot{z} &= -\frac{1}{m}\Omega_z, \end{aligned}\right\} \text{...........(4)}$$

where $\Omega = -m\left[\frac{\omega^2}{2}\left(x^2+y^2\right)+\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}+\frac{3I\mu}{2\rho_2^3}\right]$ ....(5)

$$\left.\begin{aligned}
-\frac{1}{m}\Omega_x &= \omega^2 x-\frac{(1-\mu)(x-\mu)}{\rho_1^3}-\frac{\mu(x-\mu+1)}{\rho_2^3}-\frac{9I\mu(x-\mu+1)}{2\rho_2^5},\\
-\frac{1}{m}\Omega_y &= \omega^2 y-\frac{(1-\mu)y}{\rho_1^3}-\frac{\mu y}{\rho_2^3}-\frac{9I\mu y}{2\rho_2^5},\\
\text{and}\quad -\frac{1}{m}\Omega_z &= -\frac{(1-\mu)z}{\rho_1^3}-\frac{\mu^2}{\rho_2^3}-\frac{9I\mu z}{2\rho_2^5}.
\end{aligned}\right\}\text{......(6)}$$

By Jeans law, the variation of mass of the infinitesimal body is given by

$\frac{dm}{dt} = -\alpha m^n$ i.e., $\frac{\dot{m}}{m} = -\alpha m^{n-1}$, ...(7)

where ~ is a constant coefficient and the value of exponent $n \in [0.4,\ 4.4]$ for the stars of the main sequence?

Let us introduce space time transformations as

$$\left.\begin{aligned}
&x = ¶¥^{-q},\quad y = ¦¥^{-q},\quad z = ¸¥^{-q},\quad dt = ¥^{-k}d²,\\
&°_1 = r_1¥^{-q},\quad °_2 = r_2¥^{-q},\quad ¥ = \frac{m}{m_0} < 1,
\end{aligned}\right\}\text{........ (8)}$$

where m_0 is the mass of the satellite at $t = 0$.

From Equations (7) and (8), we get

$\frac{d¥}{dt} = -\ ¥^{n-1}$, where $= \sim m_0^{n-1} = \text{constant}$ ....(9)

Following Hassan and Sweta(2017and2018),the equations of motion of the infinitesimal body in term of ξ,η,ζ as

$\xi'' - 2\omega\eta' = \Omega_\xi,\quad \eta'' + 2\omega\xi' = \Omega_\eta$ and $\zeta'' = \Omega_\zeta$,

where $\Omega(\xi,\eta,\zeta) = \frac{1}{2}\left(\frac{\alpha^2}{4}+\omega^2\right)\left(\xi^2+\eta^2\right)+\frac{(1-\mu)}{r_1}\gamma^{\frac{3}{2}}+\frac{\mu}{r_2}\gamma^{\frac{3}{2}}+\frac{3I\mu}{2r_2^3}\gamma^{\frac{5}{2}}$. ...(10)

Linear Stability of Collinear Libration Points

For collinear libration points $¦_i = ¸_i = 0$, then $L_i\left(¶_i,0,0\right)(i=1,2,3)$ are the three collinear libration points. For checking linear stability of collinear libration points, let us give slight displacement to the infinitesimal mass in the vicinity of libration points by the variations $\bar{¶}, \bar{¦}, 0$ in the coordinates of the libration points. Let $\left(¶_i+\bar{¶}, 0+\bar{¦}, 0\right)$ be displaced position of the infinitesimal mass in the plane of motion.Putting $¶ = ¶_i + \bar{¶},\ ¦ = 0 + \bar{¦},\ ¸ = 0$ then

$$¶' = \bar{¶}',\quad ¶'' = \bar{¶}'',\quad ¦' = \bar{¦}',\quad ¦'' = \bar{¦}''.$$

Thus, the variational equations

$$\bar{¶}''-2\mu\bar{¦}'=f_1\left(¶_i+\bar{¶},0+\bar{¦},0\right)\text{...(11)}$$

$$\bar{¦}''+2\mu\bar{¶}'=f_2\left(¶_i+\bar{¶},0+\bar{¦},0\right)\text{...(12)}$$

$$\bar{¸}''=f_3\left(¶_i+\bar{¶},0+\bar{¦},0\right)\text{...(13)}$$

For planar case, we consider only two equations (11) and (12)

$$\bar{¶}''-2\mu\bar{¦}'=f_1\left(¶_i+\bar{¶},0+\bar{¦}\right),$$

$$\bar{¦}''+2\mu\bar{¶}'=f_2\left(¶_i+\bar{¶},0+\bar{¦}\right).$$

$$\Rightarrow\bar{¶}''-2\mu\bar{¦}'=f_1\left(¶_i,0\right)+\bar{¶}\frac{\partial f_1}{\partial\bar{¶}}+\bar{¦}\frac{\partial f_1}{\partial\bar{¦}}+\text{higher order derivatives of } f_1\left(x,0\right)\text{at}\left(¶_i,0\right)$$

$$\text{and }\bar{¦}''-2\mu\bar{¶}'=f_2\left(¶_i,0\right)+\bar{¶}\frac{\partial f_2}{\partial\bar{¶}}+\bar{¦}\frac{\partial f_2}{\partial\bar{¦}}+\text{higher order derivatives of } f_2\left(x,0\right)\text{at}\left(¶_i,0\right)$$

But $f_1\left(¶_i,0\right)=f_2\left(¶_i,0\right)=0$ so

$$\bar{¶}''-2\mu\bar{¦}'=\bar{¶}\frac{\partial}{\partial\bar{¶}}\Omega_{\bar{¶}}+\bar{¦}\frac{\partial}{\partial\bar{¦}}\Omega_{\bar{¶}},$$

$$\bar{¦}''+2\mu\bar{¶}'=\bar{¶}\frac{\partial}{\partial\bar{¶}}\Omega_{\bar{¦}}+\bar{¦}\frac{\partial}{\partial\bar{¦}}\Omega_{\bar{¦}}.$$

Neglecting higher order terms of $\bar{¶},\bar{¦}$

$$\left.\begin{aligned}\bar{¶}''-2\mu\bar{¦}'&=\bar{¶}\,\Omega^0_{\bar{¶}\bar{¶}}+\bar{¦}\,\Omega^0_{\bar{¦}\bar{¶}},\\ \bar{¦}''+2\mu\bar{¶}'&=\bar{¶}\,\Omega^0_{\bar{¶}\bar{¦}}+\bar{¦}\,\Omega^0_{\bar{¦}\bar{¦}}.\end{aligned}\right\}\text{............ (14)}$$

Following Shahbazullah and Hassan (2014), let us write the above system of equations as

$$\left.\begin{aligned}\bar{¶}'&=0.\bar{¶}+0.\bar{¦}+1.\bar{¶}'+0.\bar{¦}',\\ \bar{¦}'&=0.\bar{¶}+0.\bar{¦}+0.\bar{¶}'+1.\bar{¦}',\\ \bar{¶}''&=\Omega^0_{\bar{¶}\bar{¶}}.\bar{¶}+\Omega^0_{\bar{¶}\bar{¦}}.\bar{¦}+0.\bar{¶}'+2\mu\bar{¦}',\\ \bar{¦}''&=\Omega^0_{\bar{¶}\bar{¦}}.\bar{¶}+\Omega^0_{\bar{¦}\bar{¦}}.\bar{¦}-2\mu\bar{¶}'+0.\bar{¦}'.\end{aligned}\right\}\text{.............(15)}$$

The system (15) can be reduced to a single matrix equation as

$$\begin{bmatrix}\bar{¶}'\\ \bar{¦}'\\ \bar{¶}''\\ \bar{¦}''\end{bmatrix}=\begin{bmatrix}0&0&1&0\\0&0&0&1\\ \Omega^0_{\bar{¶}\bar{¶}}&\Omega^0_{\bar{¶}\bar{¦}}&0&2\mu\\ \Omega^0_{\bar{¶}\bar{¦}}&\Omega^0_{\bar{¦}\bar{¦}}&-2\mu&0\end{bmatrix}\begin{bmatrix}\bar{¶}\\ \bar{¦}\\ \bar{¶}'\\ \bar{¦}'\end{bmatrix}\text{... ....(16)}$$

$$\Rightarrow X\,' = CX \text{ where } X = \begin{bmatrix} ¶ & ¦ & ¶' & ¦' \end{bmatrix}^T \text{ and } C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega^0_{¶¶} & \Omega^0_{¶¦} & 0 & 2μ \\ \Omega^0_{¶¦} & \Omega^0_{¦¦} & -2μ & 0 \end{bmatrix}$$

where $\Omega^0_{¶¶}, \Omega^0_{¶¦}, \Omega^0_{¦¦}$ are the values of $\Omega_{¶¶}, \Omega_{¶¦}, \Omega_{¦¦}\ldots$ at the libration points?

The characteristic equation of the coefficient matrix C is given by $|C - \lambda I| = 0$, where a gives Eigen values of C and *I*is the unit matrix of order 4×4.The characteristic equation can be written as

$$\begin{vmatrix} -a & 0 & 1 & 0 \\ 0 & -a & 0 & 1 \\ \Omega^0_{¶¶} & \Omega^0_{¶¦} & -a & 2μ \\ \Omega^0_{¶¦} & \Omega^0_{¦¦} & -2μ & -a \end{vmatrix} = 0$$

i.e. $a^4 - \left(\Omega^0_{¶¶} + \Omega^0_{¦¦} - 4μ^2\right)a^2 + 4\left(\Omega^0_{¶¶}\Omega^0_{¦¦} - \Omega^{0^2}_{¶¦}\right) = 0.$ ...(17)

This is the actual characteristic equation of a linear stability of the collinear libration points.Now we have to calculate $\Omega^0_{¶¶}, \Omega^0_{¶¦}$ and $\Omega^0_{¦¦}$. From Equation (4),

$$\Omega = \frac{1}{2}\left(\frac{\sim^2}{4} + μ^2\right)\left(¶^2 + ¦^2\right) + ¥^{\frac{3}{2}}\left[\frac{1-«}{r_1} + \frac{«}{r_2} + \frac{3I\,«¥}{2r_2^3}\right],$$

$$\Omega_¶ = \left(μ^2 + \frac{\sim^2}{4}\right)¶ - \frac{(1-«)\left(¶ - «\sqrt{¥}\right)}{r_1^3}¥^{\frac{3}{2}} - \frac{«\left(¶ - «\sqrt{¥} + \sqrt{¥}\right)}{r_2^3}¥^{\frac{3}{2}} + \frac{9I\,«\left(¶ - «\sqrt{¥} + \sqrt{¥}\right)}{2r_2^5}¥^{\frac{5}{2}},$$

$$\Omega_{¶¶} = \frac{\sim^2}{4} + μ^2 - ¥^{\frac{3}{2}}\left[\frac{1-«}{r_1^3} + \frac{«}{r_2^3} + \frac{9I\,«}{2r_2^5}¥\right]$$

$$+3¥^{\frac{3}{2}}\left[\frac{1-«}{r_1^5}\left(¶ - «\sqrt{¥}\right)^2 + \frac{«\left(¶ - «\sqrt{¥} + \sqrt{¥}\right)^2}{r_2^5} + \frac{15I\,«\left(¶ - «\sqrt{¥} + \sqrt{¥}\right)^2}{2r_2^7}¥\right],$$

$$\Omega_{¶¦} = 3¦¥^{\frac{3}{2}}\left[\frac{(1-«)\left(¶ - «\sqrt{¥}\right)}{r_1^5} + \frac{«\left(¶ - «\sqrt{¥} + \sqrt{¥}\right)}{r_2^5} + \frac{15I\,«\left(¶ - «\sqrt{¥} + \sqrt{¥}\right)¥}{2r_2^7}\right],$$

$$\Omega_¦ = \left(\frac{\sim^2}{4} + μ^2\right)¦ - \frac{(1-«)¦}{r_1^3}¥^{\frac{3}{2}} - \frac{«¦}{r_2^3}¥^{\frac{3}{2}} - \frac{15I\,«¦}{2r_2^5}¥^{\frac{5}{2}},$$

$$\Omega_{¦¦} = \frac{\sim^2}{4} + μ^2 - ¥^{\frac{3}{2}}\left[\frac{1-«}{r_1^3} + \frac{«}{r_2^3} + \frac{9I\,«¥}{2r_1^5}\right] + 3¥^{\frac{3}{2}}\left[\frac{1-«}{r_1^5} + \frac{«}{r_2^5} + \frac{15I\,«¥}{2r_2^7}\right].$$

At the libration points,

$$\Omega^0_{¶¶} = \frac{\sim^2}{4} + µ^2 + \frac{(3¥-1)(1-«)}{|x-«|^3} + \frac{(3¥-1)«}{|x-«+1|^3} + \frac{(5¥-1)9I«}{2|x-«+1|^5},$$

$$\Omega^0_{¶¦} = \frac{3¦}{\sqrt{¥}}\left[\frac{1-«}{|x-«|^4} + \frac{«}{(x-«+1)^4} + \frac{15I«}{2|x-«+1|^6}\right] = 0, \quad (\text{where } ¦ = 0)$$

$$\Omega^0_{¦¦} = \left(\frac{\sim^2}{4} + µ^2\right) + \left[\frac{3}{¥(x-«)^2} - 1\right]\frac{1-«}{|x-«|^3} + \left[\frac{3}{¥(x-«+1)^2} - 1\right]\frac{«}{|x-«+1|^3}$$

$$+\left[\frac{5}{¥(x-«+1)^2} - 1\right]\frac{9I«}{2|x-«+1|^5}.$$

At the libration point $L_1(¶_1, 0, 0)$

$$\Omega^0_{¶¶} = \frac{\sim^2}{4} + µ^2 + \frac{(3¥-1)(1-«)}{|a_1-«|^3} + \frac{(3¥-1)«}{|a_1-«+1|^3} + \frac{(5¥-1)9I«}{2|a_1-«+1|^5},$$

$$\Omega^0_{¶¦} = \frac{3¦}{\sqrt{¥}}\left[\frac{1-«}{|a_1-«|^4} + \frac{«}{(a_1-«+1)^4} + \frac{15I«}{2|a_1-«+1|^6}\right],$$

$$\Omega^0_{¦¦} = \left(\frac{\sim^2}{4} + µ^2\right) + \left[\frac{3}{¥(a_1-«)^2} - 1\right]\frac{1-«}{|a_1-«|^3} + \left[\frac{3}{¥(a_1-«+1)^2} - 1\right]\frac{«}{|a_1-«+1|^3}$$

$$+\left[\frac{5}{¥(a_1-«+1)^2} - 1\right]\frac{9I«}{2|a_1-«+1|^5}.$$

But $¶_1 < « - 1$ so taking $¶_1 = « - 1 - °$ where $°$ is very small quantity [Hassan et al. (2017)]

$$\therefore ¶_1 - « + 1 = -° < 0 \text{ and } ¶_1 - « = -1 - ° < 0,$$

$$|¶_1 - « + 1| = -(¶_1 - « + 1) = -(-°) = °,$$

$$|¶_1 - «| = -(¶_1 - «) = -(-1-°) = 1 + °.$$

$$\Omega^0_{¶¶} = \frac{\sim^2}{4} + µ^2 + \frac{(3¥-1)(1-«)}{(1+°)^3} + \frac{(3¥-1)«}{°^3} + \frac{(5¥-1)9I«}{2°^5},$$

$$= \frac{\sim^2}{4} + µ^2 + (3¥-1)(1-«)(1-3°) + \frac{(3¥-1)«}{°^3} + \frac{(5¥-1)9I«}{2°^5},$$

$$\left.\begin{aligned}\Omega^0_{¶¶} &= \frac{\sim^2}{4}+μ^2+\frac{(3¥-1)(1-«)}{(1+°)^3}+\frac{(3¥-1)«}{°^3}+\frac{(5¥-1)9I«}{2°^5},\\ &= \frac{\sim^2}{4}+μ^2+(3¥-1)(1-«)(1-3°)+\frac{(3¥-1)«}{°^3}+\frac{(5¥-1)9I«}{2°^5},\end{aligned}\right\} \quad ....(18)$$

$$\Omega^0_{¶¦} = \frac{3¦}{\sqrt{¥}}\left[\frac{1-«}{(1+°)^4}+\frac{«}{°^4}+\frac{15I«}{2°^6}\right], \quad ....(19)$$

$$\Omega^0_{¦¦} = \left(\frac{\sim^2}{4}+μ^2\right)+\left[\frac{3}{¥}(1-2°)-1\right](1-«)(1-3°)+\left[\frac{3}{¥°^2}-1\right]\frac{«}{°^3}+\left[\frac{5}{¥°^2}-1\right]\frac{9I«}{2°^5}. \quad .....(20)$$

The characteristic equation (17) is a biquadratic equation in a and quadratic equation in a^2, so let a_1^2 and a_2^2 be two roots of the characteristic equation (17) then the libration points are stable if a_1^2 and a_2^2 are real and negative, hence

$(\text{i})\ a_1^2 + a_2^2 < 0,$

$(\text{ii})\ a_1^2 . a_2^2 > 0,$

(iii) Discriminant of $(17) \geq 0.$

$$\left.\begin{aligned}&\text{i.e. } \Omega^0_{¶¶}+\Omega^0_{¦¦}-4μ^2<0,\\ &\Omega^0_{¶¶}.\Omega^0_{¦¦}>0,\\ &D=\left(\Omega^0_{¶¶}+\Omega^0_{¦¦}-4μ^2\right)-4\Omega^0_{¶¶}.\Omega^0_{¦¦}\geq 0.\end{aligned}\right\} \quad ....(21)$$

Table 1:Characteristic roots and the nature of Stability of Libration Points $L_1(¶_1,0,0)$ at $« = 0.019,\ \sim = 0.1,\ I = 0.01$

Sl. No.	Libration Points $(¶_i, ¦_i)$	¥	Corresponding Characteristics roots (a_1, a_2, a_3, a_4)	Nature
01	-1.1774738957, 0	1	-2.1053543968, -1.831555131131 I, 1.831555131131 I, 2.1053543968	Unstable
02	-1.2050310372, 0	0.98	-205152719289, -1.7322075747 I, - 1.7322075747 I, 2.5152719289	Unstable
03	-1.1907779084, 0	0.96	-2.5295433630, -1.7287759247 I, 1.7287759247 I, 2.5295433630	Unstable
04	-1.1764131583, 0	0.94	-2.5442123837, -1.7253037110 I, 1.7253037110 I, 2.5442123837	Unstable
05	-1.1619332697, 0	0.92	-2.5592967623, -1.7217892553 I, - 1.7217892553 I, 2.5592967623	Unstable
06	-1.1473345333, 0	0.90	-2.5748154352, - 1.7182307502 I, 1.7182307502 I, 2.5748154352	Unstable
07	-0.6193146504, 0	0.30	-3.4494685493, 3.4494685493 -1.5673916332 I, -1.5673916332 I	Unstable

For the second collinear libration point $L_2(\xi_2, 0, 0)$, putting $x = \xi_2 = \mu - 1 + \rho$ in Equation (20), (21) and (22), we get

$$\Omega^0_{\xi\xi} = \frac{\alpha^2}{4} + \omega^2 + (3\gamma - 1)(1-\mu)(1+3\rho) + \frac{(3\gamma-1)\mu}{\rho^3} + \frac{(5\gamma-1)9I\mu}{2\rho^5},$$

$$\Omega^0_{\xi\eta} = 0,$$

$$\Omega^0_{\eta\eta} = \left(\frac{\alpha^2}{4} + \omega^2\right) + \left[\frac{3}{\gamma}(1+2\rho) - 1\right](1-\mu)(1+3\rho) + \left[\frac{3}{\gamma\rho^2} - 1\right]\frac{\mu}{\rho^3} + \left[\frac{5}{\gamma\rho^2} - 1\right]\frac{9I\mu}{2\rho^5}.$$

Table 2: Characteristics roots and the nature of Stability of Libration Points $L_2(\xi_2, 0, 0)$ at $\mu = 0.019,\ \alpha = 0.1,\ I = 0.01$

Sl. No.	Libration Points (ξ_i, η_i)	γ	Corresponding Characteristics roots $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	Nature
01	-0.8072796446, 0	1	-3.0048759467, - 2.3803813971 I,+ 2.3803813971 I, 3.0048759467	Unstable
02	-0.7583906198, 0	0.98	-3.6141621541, -2.3874371587 I, 23.3874371587 I, 3.6141621541	Unstable
03	-0.7506032963, 0	0.96	-3.5710843468, -2.3539051234 I, 2.3539051234 I, 3.5710843468	Unstable
04	-0.7427187900, 0	0.94	-3.5272211995, -2.3199812774 I, 2.3199812774 I, 3.5272211995	Unstable
05	-0.7347337236, 0	0.92	-3.4825437656, -2.2856514618 I, 2.2856514618 I, 3.4825437656	Unstable
06	-0.7266445240	0.90	-3.4370215270, -2.2509005457 I, 2.2509005457 I, 3.4370215270	Unstable
07	-0.4012798775, 0	0.3	-1.3130278625, 1.3130278625, -0.6416592431 I, 0.6416592431 I	Unstable

For the third collinear libration point $L_3(\xi_3, 0, 0)$, putting $x = \xi_3 = \mu + 1 - \rho$ in Equations (23), (24) and (25), we get

$$\Omega^0_{\xi\xi} = \frac{\alpha^2}{4} + \omega^2 + (3\gamma-1)(1-\mu)(1+3\rho) + \frac{(3\gamma-1)\mu}{8}\left(1+\frac{3}{2}\rho\right) + \frac{(5\gamma-1)9I\mu}{64}\left(1+\frac{5}{2}\rho\right),$$

$$\Omega^0_{\eta\eta} = \left(\frac{\alpha^2}{4} + \omega^2\right) + \left[\frac{3(1+2\rho)}{\gamma} - 1\right](1-\mu)(1+3\rho) + \left[\frac{3(1+\rho)}{\gamma} - 1\right]\frac{\mu}{8}\left(1+\frac{3}{2}\rho\right)$$
$$+ \left[\frac{5(1+\rho)}{\gamma} - 1\right]\frac{9I\mu}{64}\left(1+\frac{5}{2}\rho\right).$$

Table 3: Characteristics roots and the nature of Stability of Libration Points $L_3(\xi_3, 0, 0)$ at $\mu = 0.019,\ \alpha = 0.1,\ I = 0.01$

Sl. No.	Libration Points (ξ_i, η_i)	γ	Corresponding Characteristics roots $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	Nature
01	1.0079162896, 0	1	-0.2219771557, -1.0161214989 I, 1.061214989 I, 0.2219771557	Unstable
02	0.9888160228, 0	0.98	-0.1451311550, 0.1451311550, - 1.000400519 I, 1.0004400519 I	Unstable
03	0.9753537655, 0	0.96	-0.1078098455 I, 0.178098455 I, - 0.9785778373 I, 0.9785778373 I	Stable
04	0.9617973307, 0	0.94	-0.2190311213 I, 0.2190311213 I, - 0.9542378331 I, 0.954237833 I	Stable
05	0.9481440321, 0	0.92	-0.2986854803 I, 0.2986854803 I, - 0.9265642533 I, 0.9265642533 I	Stable
06	0.9343910468, 0	0.90	-0.3707261219 I, 0.3707261219 I, - 0.8940723451 I, 0.8940723451	Unstable

Linear Stability of Triangular Libration Points

For checking linear stability of triangular points, let us give slight displacement to the infinitesimal mass from $L_4(\xi_4, \eta_4, 0)$ by the small increments $\bar{\xi}, \bar{\eta}, 0$, then the co-ordinates of the displaced position of the infinitesimal mass is $(\xi_4 + \bar{\xi}, \eta_4 + \bar{\eta}, 0)$,

where $\xi_4 = « -\frac{3}{2} - 8I + \frac{\sim^2}{3}$ and $\eta_4 = \frac{\sqrt{3}}{2}\left[\frac{5}{3} + \frac{14I}{3} - \frac{4}{9}\sim^2\right]$

Since at the triangular points $\Omega_{\xi\eta} \neq 0$, hence for the triangular points the characteristic equation is

$$a^4 - \left(\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} - 4\mu^2\right)a^2 + \left[\Omega^0_{\xi\xi}.\Omega^0_{\eta\eta} - \left(\Omega^0_{\xi\eta}\right)^2\right] = 0 \text{ ....(22)}$$

The conditions of stability are

$(\text{i})\,\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} - 4\mu^2 < 0,$

$(\text{ii})\,\Omega^0_{\xi\xi}.\Omega^0_{\eta\eta} - \left(\Omega^0_{\xi\eta}\right)^2 > 0,$

$(\text{iii})\,D = \left[\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} - 4\mu^2\right]^2 - 4\left[\Omega^0_{\xi\xi}.\Omega^0_{\eta\eta} - \left(\Omega^0_{\xi\eta}\right)^2\right] \geq 0.$

Now let us find the values $\Omega^0_{\xi\xi}, \Omega^0_{\xi\eta}, \Omega^0_{\eta\eta}$ of at $L_4(\xi_4, \eta_4, 0)$. From Equation (4),

$$\Omega = \frac{1}{2}\left(\frac{\alpha^2}{4} + \omega^2\right) + \gamma^{\frac{3}{2}}\left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{3I\mu\gamma}{2r_1^3}\right),$$

$$\Omega_{\xi} = \left(\frac{\alpha^2}{4} + \omega^2\right)\xi - \gamma^{\frac{3}{2}}\left[\frac{(1-\mu)\left(\xi - \mu\sqrt{\gamma}\right)}{r_1^3} + \frac{\mu\left(\xi - \mu\sqrt{\gamma} + \sqrt{\gamma}\right)}{r_2^3} + \frac{9I\,\mu\left(\xi - \mu\sqrt{\gamma} + \sqrt{\gamma}\right)}{2r_1^5}\right]$$

$$\Omega_{\xi\xi} = \left(\frac{\alpha^2}{4} + \omega^2\right) - \gamma^{\frac{3}{2}}\left[\frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} + \frac{9I\mu\gamma}{2r_2^5}\right]$$
$$+ 3\gamma^{\frac{3}{2}}\left[\frac{(1-\mu)\left(\xi - \mu\sqrt{\gamma}\right)}{r_1^5} + \frac{\mu\left(\xi - \mu\sqrt{\gamma} + \sqrt{\gamma}\right)^2}{r_2^5} + \frac{15I\mu\gamma\left(\xi - \mu\sqrt{\gamma} + \sqrt{\gamma}\right)}{2r_2^7}\right],$$

$$\Omega_{\xi\eta} = 3\eta\gamma^{\frac{3}{2}}\left[\frac{(1-\mu)\left(\xi - \mu\sqrt{\gamma}\right)}{r_1^5} + \frac{\mu\left(\xi - \mu\sqrt{\gamma} + \sqrt{\gamma}\right)}{r_2^5} + \frac{9I\mu\gamma\left(\xi - \mu\sqrt{\gamma} + \sqrt{\gamma}\right)}{2r_2^7}\right],$$

$$\Omega_{\eta} = \left(\frac{\alpha^2}{4} + \omega^2\right)\eta - \gamma^{\frac{3}{2}}\left[\frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} + \frac{9I\mu\gamma}{2r_2^5}\right]\eta,$$

$$\Omega_{\eta\eta} = \left(\frac{\alpha^2}{4} + \omega^2\right)\eta - \gamma^{\frac{3}{2}}\left[\frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} + \frac{9I\mu\gamma}{2r_2^5}\right] + 3\eta^2\gamma^{\frac{3}{2}}\left[\frac{(1-\mu)}{r_1^5} + \frac{\mu}{r_2^5} + \frac{15I\mu\gamma}{2r_2^7}\right],$$

$$\Omega_{\xi\xi}=\left(\frac{\alpha^2}{4}+\omega^2\right)-\left[\frac{(1-\mu)}{\rho_1^3}+\frac{\mu}{\rho_2^3}+\frac{9I\mu}{2\rho_2^5}\right]$$

$$+3\left[\frac{(1-\mu)(x-\mu)^2}{\rho_1^5}+\frac{\mu(x-\mu+1)^2}{\rho_2^5}+\frac{15I\mu(x-\mu+1)^2}{2\rho_2^5}\right],$$

$$\Omega_{\xi\eta}=3y\left[\frac{(1-\mu)(x-\mu)}{\rho_1^5}+\frac{\mu(x-\mu+1)}{\rho_2^5}+\frac{15I\mu(x-\mu+1)}{2\rho_2^5}\right],$$

$$\Omega_{\eta\eta}=\left(\frac{\alpha^2}{4}+\omega^2\right)-\left[\frac{(1-\mu)}{\rho_1^3}+\frac{\mu}{\rho_2^3}+\frac{9I\mu}{2\rho_2^5}\right]+3y^2\left[\frac{(1-\mu)}{\rho_1^5}+\frac{\mu}{\rho_2^5}+\frac{15I\mu}{2\rho_2^7}\right].$$

At $L_4(\xi_4,\eta_4,0)$,

$$\rho_1^2=(x-\mu)^2+y^2=(\xi_4-\mu)^2+\eta_4^2,$$

$$\rho_1=\left[\frac{13}{3}+\frac{107}{3}I+\frac{241}{3}I^2-\frac{19}{9}\alpha^2-\frac{76}{9}I\alpha^2+\frac{7}{27}\alpha^4\right]^{\frac{1}{2}}$$

$$\rho_2^2=(x-\mu+1)^2+y^2=(\xi_4-\mu+1)^2+\eta_4^2,$$

$$\rho_2=\left[\frac{7}{3}+\frac{59}{3}I+\frac{241}{3}I^2-\frac{49}{27}\alpha^2-\frac{256}{27}I\alpha^2+\frac{25}{81}\alpha^4\right]^{\frac{1}{2}}.$$

$$\Omega_{\xi\xi}^0=\left(\frac{\alpha^2}{4}+\omega^2\right)-\left(\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}+\frac{9I\mu}{2\rho_2^5}\right)+3\left[\frac{1-\mu}{\rho_1^5}\left(\frac{9}{4}+24I+64I^2-\alpha^2-\frac{16}{3}I\alpha^2+\frac{\alpha^4}{9}\right)\right.$$

$$\left.+\mu\left(\frac{1}{\rho_2^5}+\frac{15I}{2\rho_2^7}\right)\left(\frac{1}{4}+8I+64I^2-\frac{\alpha^2}{3}-\frac{16I\alpha^2}{3}+\frac{\alpha^4}{9}\right)\right],$$

$$\Omega_{\xi\eta}^0=\frac{3\sqrt{3}}{2}\left(\frac{5}{3}+\frac{14}{3}I-\frac{4}{9}\alpha^2\right)\left[\frac{1-\mu}{\rho_1^5}\left(-\frac{3}{2}-8I+\frac{\alpha^2}{3}\right)+\frac{\mu}{\rho_2^5}\left(1+\frac{15}{2\rho_2^2}I\right)\left(-\frac{1}{2}-8I+\frac{\alpha^2}{3}\right)\right],$$

$$\Omega_{\eta\eta}^0=\left(\frac{\alpha^2}{4}+\omega^2\right)-\left(\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}+\frac{9I\mu}{2\rho_2^5}\right)$$

$$+\left(\frac{1-\mu}{\rho_1^5}+\frac{\mu}{\rho_2^5}+\frac{15I\mu}{2\rho_2^7}\right)\left(\frac{25}{4}+35I+49I^2-\frac{10}{3}\alpha^2-\frac{28}{3}I\alpha^2+\frac{4}{9}\alpha^4\right).$$

Table 4: Characteristics roots and the nature of Stability of Libration Points $L_4(¶_4, ¦_4, 0)$ **at** $« = 0.019,\ \sim = 0.1,\ I = 0.01$

Sl. No.	Libration Points $(¶_j, ¦_j)$	¥	Corresponding Characteristics roots (a_1, a_2, a_3, a_4) Ta	Nature
01	-0.4810000000, 0.8660254037	1	- 0.3841856261 I, 0.3841856261 I, -0.9232558717 I, 0.9232558717 I	Stable
02	-0.5389656752, 0.8092034601	0.98	-0.4500937607 I, 0.4500937607 I, -0.8959998035 I, 0.8959998035 I	Stable
03	-0.594978356, 0.7524504290	0.96	-0.548742729 I, 0.5408742729 I, -0.8371053331 I, 0.8371053331 I	Stable
04	-0.6345868437, 0.7017950925	0.94	-0.1098173701 - 0.7121545814 I, -0.1098173701 + 0.7121545814 I, 0.1098173701 - 0.7121545814 I, 0.1098173701 + 0.7121545814 I	Unstable
05	-0.6615428294, 0.6578007288	0.92	-0.2158125430 - 0.7317227573 I, -0.2158125430 + 0.7317227573 I, 0.2158125430 - 0.7317227573 I, 0.2158125430 + 0.7317227573 I,	Unstable
06	-0.6794608921, 0.6195984520	0.90	-0.2847832526 - 0.7500773410 I, -0.2847832526 + 0.7500773410 I, 0.2847832526 - 0.7500773410 I, 0.2847832526 + 0.7500773410 I	Unstable

Table 5: Characteristics Roots and the Nature of Stability of Libration Points $L_5(¶_4, -¦_4, 0)$ **at** $« = 0.019,\ \sim = 0.1,\ I = 0.01$

Sl. No.	Libration Points $(¶_j, ¦_j)$	¥	Corresponding Characteristics roots (a_1, a_2, a_3, a_4)	Nature
01	-0.4810000000, -0.8660254037	1	-0.3841856261 I, 0.38418562616 I, -0.9232558717 I, 0.9232558717 I	Stable
02	-0.5389656752, -0.8092034601	0.98	-0.4500937607 I, 0.4500937607 I, -0.8959998035 I, 0.8959998035 I	Stable
03	-0.5949978356, -0.7524504290	0.96	-0.5468742729 I, 0.5468742729 I, -0.8371053331 I, 0.8371053331 I	Stable
04	-0.6345868437, -0.7017950925	0.94	-0.1098173701 - 0.7121545814 I, -0.1098173701 + 0.7121545814 I, 0.1098173701 - 0.7121545814 I, 0.1098173701 + 0.7121545814 I	Unstable
05	-0.6615428294, -0.6578007288	0.92	-0.2158125430 - 0.7317227573I, -0.2158125430 + 0.7317227573 I, 0.2158125430 - 0.7317227573 I, 0.2158125430 + 0.7317227573 I,	Unstable
06	-0.6794608921, -0.6195984520	0.90	-0.2847832526 - 0.7500773410 I, -0.2847832526 + 0.7500773410 I, 0.2847832526 - 0.7500773410 I, 0.2847832526 + 0.7500773410 I	Unstable
07	-0.4040782891, -0.0803775708	0.3	-0.8227986832 - 0.5779753962 I, -0.8227986832 + 0.5779753962 I, 0.8227986832 - 0.5779753962 I, 0.8227986832 + 0.5779753962 I,	Unstable

Discussions and Conclusions

From the Tables 1 and 2, it is clear that all the collinear libration points $L_i\ (i = 1, 2)$ are unstable for $« = 0.019, \sim = 0.1, I = 0.01$ and for all values of ¥ when $0.9 \le ¥ \le 1$ but from Table 3, it is clear that L_3 is unstable for $¥ \in [0.98, 1]$ and stable for $¥ \in [0.92, 0.96]$. From Tables 4 and 5, it is clear that the triangular libration points L_4 and L_5 are stable only for the values of $¥ \in [0.96, 1]$ and unstable for $¥ \le 0.94$.

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