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SOME STUDY OF ISHIKAWA ITERATION PROCESS IN Lp SPACE

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ABSTRACT

In this paper we have discussed Ishikawa Iteration Process in Lp space which are of such importance in solving the problems related to economic and game theory. It is our purpose in this paper to prove convergence theorems for both the Isikawa Iteration Process and the Mann Iteration Process for continuous quasi-contractive mapping in Lp space for 1≤p≤q.

KEYWORDS: *Convergence, quasi-Contractive Mapping, Lp Space. Iteration, Importance.*

Introduction

Our method will, in addition, show that the compactness assumption on K as given by childume [1] and Rohades [2], is not need. For particular choices of the real sequences $(\alpha_n), (\beta_n)$ and (C_n) explicit convergence rates are calculated which, for p = 2, agree with results. The results of this chapter, together with earlier results of the authors childune [3] and [4], then show that either the Ishikawa or Mann iteration process can be used to approximate the fixed point of a continuous quasi-contractive mappings in L_p or L_p space. 1 $\leq$ p $\leq$ ∞ In all cases, no compactness assumption is needed.

Definition 1 : Qussi Contractive Mapping

Let K be a subset of Hilbert space H.A mapping T : K $\rightarrow$ K into itself is called quasi-contractive if there exists a constant K $\in$ [0,1] such that for each x, y $\in$ k, $\|T_x - T_y\|$ $\leq K \max\{\|x-y\| \; \|x-T_x\|, \|y-T_y\|, \|x-T_y\|, \|y-T_x\|\}$ (1)

Definition 2 : Ishikawa Iteration Process

The Ishikawa Iteration Process is defined as follows : For K a convex subset of a Banach space, X, and T a mapping of K into itself, the sequence $\{X_n\}_n^{\infty}$ = 0 in K is defined by

$$X_o \in k$$

$$X_{n+1} = (1-\alpha_n)X_n + \alpha_n T\,[\beta_n T_{Xn} + (1-\beta_n)X_n], n \geq 0$$

Where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ satisfy 0 $\leq \alpha_n, \leq \; \beta_n \leq 1$

For all n, $\lim_{n\to\infty n} \beta = 0$ and

$$\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty \qquad [A\}$$

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Definition 3: The Mann iteration Process

This process is defined as follows with K, X and T as in [A], the sequence $\{X_n\}_{n=0}^{\infty}$ is k is defined by

$$X_o \in k$$

$$X_{n+1} = (1 - C_n)X_n + C_n T_{X_n}, n \geq 0.$$

Where $\{C_n\}_{n=0}^{\infty}$ satisfied : $0 \leq C_n \leq 1$ for all n,

$$\lim_{n=\alpha} C_n = 0 \; and \sum_{n=0}^{\infty} C_n = \infty \qquad [B]$$

In some application the condition

$$\sum_{n=0}^{\infty} C_n = \infty$$

Is replaced by condition

$$\sum_{n=0}^{\infty} C_n(1 - C_n = \infty$$

The iteration methods [A] and [B] have been studied by several authors and have been used to approximate fixed point of several nonlinear mapping. Although the iteration process [A] by using β_n = 0 for all a and putting different conditions a_n examples given by Rhoades show that the two iteration methods may exhibit different behaviours for different classes of nonlinear mappings. Rhoades proved that most of the results established by Rhoades, using the Mann iteration process can be extended to the Ishikawa iteration scheme, hence providing a much larger clap of fixed point iteration procedures. Rhoades also noted that the Mann iteration process can also be used to approximate fixed points of quasi-contractive mappings. He then posed the following question which remained open for many years : can the Mann iteration process be replaced by that of Ishikawa for continuous quasi-contractive mapping of K into itself, where K is compact convex subset of Hilbert space This question has recently been resolved in the affirmative by the author. The author, in fact, proved that the Ishikawa iteration scheme converges strongly to fixed points of continuous quasi contractive mapping in L_p or l_P spaces $p \geq 2$, if $k^2 < (p-1)^{-1}$ where K is the constant appearing in inequality (1). The special case p=2 then resolved the question raised by Rhoades. The author proved that the Mann iteration process can also be used for continuous quasi-contractive mapping in L_p spaces for $2 < p < \infty$. The method used earlier [6,9] unfortunately could not be modified to provide any convergence theorem for either the Ishikawa process of the Mann process for continuous quasi-contractive mapping in L_p space when $1 \leq p \leq 2$.

Let x be a Banach space, we shall denote by J the normalized duality mapping for x to 2^{x^*} given by

Jx = { f* $\varepsilon\, x^* : \| f^* \|^2 = \| x \|^2 = < x, f^* > \} \|$

Where x* denotes the dual space of x and < > denotes the generalized duality pairing. If x* is strictly convex, then J is single- values and if x* is uniformly convex, then J is uniformly continuous bounded sets.

Theorem (a) : Let j denote the single- valued normalized duality map on x and x = L_p, $1 \leq p \leq 2$. Then each x, y in x, the following in quality holds.

(p-1) $\| x + y \|^2 \leq \| x \|^2 + \| y \|^2 + 2 < x, j(y) >$ (A1)

Remark 1 : Rich [30] proved that if x* is uniformly convex then there exists a continuous non decreasing function.

a : $[0, \infty) \to [0, \infty)$

such that

a(o), o, a (ct) $\leq$ ca (f) for all o $\geq$ 1

and

$$\| x + y \|^2 \leq \| x \|^2 + 2 < y, j(x) > + \max\{\| x \|, 1\} \| y \| b \| y \|)\}$$

for all x, y εX (2)

Lemna 1 : for any real number λ > o, the inequality :

$$\| \lambda x + (1 - \lambda) y - z \|^2 \leq (1 - \lambda) \| y - z \|^2 + \lambda \| x - z \|^2 - (p - 1)\lambda \| x - y \|^2$$

$$+ \max\{\| y - z \|, 1\} \lambda \| x - y \| b \lambda \| x - y \|)\}$$

Holds for all x, y, z ε X, where x = L_p, 1 $\leq$ p $\leq$ 2.

Proof : By using inequality (2)

$$\| x + (1 - \lambda) y - z \|^2 = \| (y - z) + \lambda(x - y) \|^2$$

$$\leq \| y - z \|^2 + 2\lambda \leq x - y, j(y - z) >$$

$$+ \max\{\| y - z \|, 1\} \lambda \| x - y \| b \lambda \| x - y \|)\}$$

$$= \| y - z \|^2 + 2\lambda \leq x - z, j(y - z) > -2\lambda \| y - z \|^2$$

$$\max[\| y - z \|, 1] \lambda \| x - y \| b(\lambda \| x - y \|) \quad \text{(L1.1)}$$

Now, by inequality (A_1)

$$\| y - z \|^2 = \| (z - x) + (y - z) \|^2$$

$$< \left(\frac{1}{p-1}\right) [\| z - x \|^2 + \| y - z \|^2 + 2 \leq x - x, j(y - z) >]$$

So that

(p-1) $\| y - z \|^2 \leq \| z - x \|^2 + \| y - z \|^2 - 2 < x - z, (y - z) >$ or

2 $\leq$ x- (z, j) (y-z) > $\leq \| z - x \|^2 + \| y - z \|^2 - (p - 1) \| y - z \|^2$ (L1.2)

Substitution of (L1.2) in (L1.1) yields

$$\| \lambda x + 1(1 - \lambda) y - z \|^2 \leq \| y - z \|^2 + \lambda \| z - x \|^2 + \| y - z \|^2$$

$$- (p-1)\lambda \| y - x \|^2 - 2\lambda \| y - z \|^2 + \max(\| y - z \|, 1\}$$

$$\| x - y \| b(\lambda \| x - y \|)$$

Which simplifies to give

$$\| \lambda x + (1 - \lambda) y - z \|^2 \leq \| 1 - \lambda \| \| y - z \|^2 + \lambda \| x - z \|^2 - \|$$

$$(p - 1)\lambda \| y - x \|^2$$

$$+ \max(\| y - z \|, 1\} \lambda \| x - y \| b(\lambda \| x - y \|)$$

As required.

Theorem 1: Suppose K be a closed convex and bounded subset of X and T be a continuous quasi contractive mapping of K into itself with $K^2 \leq$ (P-1). Where K ε(0,1) is the constant appearing in inequality (1), where X = Lp, 1 $\leq$ P$\leq$ 2. (so let $\{\lambda_n\}_{n=0}^{\infty}$ be a real sequence satisfying :

O $\leq \lambda_n \leq$ 1 : for all n;

$$\sum_{n=0}^{\infty} \lambda_n = \infty \quad and$$

$$\sum_{n=0}^{\infty} \lambda_n p \leq \infty$$

Then the sequence $\{X_n\}_{n=0}^{\infty}$ defined iteratively by $X_o \, \varepsilon \, K$

$$X_{n+1} = (1-\lambda_n) X_n + \lambda_n TX_n, \; n \geq 0 \qquad (1.1)$$

Converges strongly to the unique fixed point of T in K.

Proof: The existence of a unique fixed point for T in K follows from ciric [10]. Let q denote the fixed point of T in K. Let y = q in inequality (1) to obtain

$$\| T_X - q \| \leq k \, max\{ \|x-q\|, \|x-T_x\|\} \qquad (1.2)$$

For all x ε K. from equation (1.1) we have

$$\| X_{n+1} - q \|^2 = \| (1-\lambda_n) + \lambda_n - q \|^2$$
$$\leq (1-\lambda_n) \| X_n - q \|^2 + \lambda_n \| T_{Xn} - q \|^2 - (p-1)\lambda_n \| X_n - TX_n \|^2$$
$$+ \max[\| X_n - q \|, 1] \lambda_n \| X_n - TX_n \| \, b(\lambda_n \| X_n - TX_n \|)$$
$$\leq (1-\lambda_n [\| X_n - q \|^2 + \lambda_n \| T_{Xn} - q \|^2 - (p-1) \lambda_n \| X_n - TX_n \|^2$$
$$+ \max[\| X_n - q \|, 1] \lambda_n \| X_n - TX_n \| \max[\| X_n - TX_n \|, 1] \, b(\lambda_n)$$

So that

$$\| X_{n+1} - q \|^2 \leq (1-\lambda_n) [\| X_n - q \|^2 + \lambda_n \| T_{Xn} - q \|^2$$
$$-(p-1) \lambda_n \| X_n - TX_n \|^2 + M \lambda_n \, b(\lambda_n) \qquad (1.3)$$

For some constant M > o since K is bounded. Observe that since T is quasi-contractive, it follows from (1.2) that

$$\| TX_n - q \| \leq k \max[\| X_n - q \|, \| X_n - TX_n \|].$$

Case 1. We consider the set of all positive integers n such that

$\| T_{Xn} - q \| \leq k \| X_n - q \|$. In this case inequality (1.3) yields

$$\| X_{n+1} - q \|^2 \leq \| (1-\lambda_n) \| + \| X_n - q \|^2 + \lambda_n k^2 \| X_n - q \|^2 + M \lambda_n \, b(\lambda_n)$$

$$= (1-\lambda_n + \lambda_n k^2) \| X_n - q \|^2 + M \lambda_n \, b(\lambda_n) \qquad (1.4)$$

Case 2 We consider the set of all positive integers n such that

$\| T_{Xn} - q \| \leq k \| X_n - TX_n \|$. In this case we obtain from inequality (1.3)

$\| X_{n+1} - q \|^2 \leq (1-\lambda_n) [\| X_n - q \|^2 + \lambda_n k^2 \| X_n - T_{Xn} \|^2$

$-$ (p-1) $\lambda_n \| X_n - TX_n \|^2 +$ M $n^b(\lambda_n)$

$= (1- \lambda_n) \| qx_n - q \|^2 - \lambda_n [(p-1) - k^2] \| X_n - T_{Xn} \|^2 + M\lambda_n b(\lambda_n)$

$\leq (1-\lambda_n) \| X_n - q \|^2 M\lambda_n b(\lambda_n)$

Since $K^2 \leq p - 1,$ (1.5)

From (1.4) and (1.5) we obtain that for all positive integers n,

$\| X_{n+1} - q \|^2 \leq [(1-\lambda_n)(1 - k^2)] [\| X_n - q \|^2 + M\lambda_n b(\lambda_n)$ (1.6)

Set $\rho_n = \| X_n - q \|^2$, $\lambda_n = (1-k^2)$n and $\sigma_n = M\lambda_n b(\lambda_n)$ to obtain from (1.6) that

$\rho_{n+1} \leq (1-\lambda_n) \rho_n + \sigma_n$.

Condition (ii) and (iii) imply that

$$\sum_{n=0}^{\infty} \sigma_n = M \sum_{n=0}^{\infty} \lambda_n b(\lambda_n) \leq \infty$$

So by lemna 2 $\rho_n \to 0$ as n $\to \infty$, which implies $X_n \to q$ as n $\to \infty$. This completes proof of Theorem 1.

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